1. Introduction

1.1 The problem

The context for this paper is the interaction between a firm that produces and processes durable pollution[1] and a regulator charged with the tasks of (a) employing a mandated technology to process the public stock of pollution, and (b) designing a contract that induces the firm to adopt a socially desirable technology for processing its private pollution stock. We represent the firm's pollution-related technology in terms of easily measurable statistical parameters. Although this paper is theoretical in nature, our sparse and direct, yet flexible, conceptualization of the relevant aspects of the firm should allow straightforward application of our results to concrete situations.

Given our focus on durable pollution, we postulate the existence of a private pollution stock and a public pollution stock. We adopt the view that pollution becomes a matter of social concern only when it is emitted from the firm's private stock into the regulator's public stock. Since the firm is the only producer of pollutants, the public pollution stock consists entirely of the accumulated emissions from the firm. In our model, the firm bears costs associated with pollution accumulation in its private stock and emission into the public stock. Consequently, it must operate simultaneously on two margins: (1) the production of pollutants, and (2) the emission of pollutants. We model the firm's decision-making on the second margin taking its decisions regarding production as exogenously given.

Consider an infinitely-lived firm, with discount rate $\alpha \in A$, that creates durable pollution jointly with a private good. At every instant, the firm chooses the output of the private good and pollution by selecting the activity level of its production process. The resulting pollution flows into the firm's private stock of pollutants.[2] At each moment, the firm can deal with a unit of private pollution stock in one of two ways: the firm

[1] This refers to pollutants that can accumulate over time. As our model allows pollutants to decay at arbitrary positive real rates, the only pollutants left out of its ambit are those with an infinite rate of decay, i.e., non-durable pollutants that dissipate instantaneously.

[2] For example, holding tanks for chemical wastes, fly-ash dumps at thermal power plants, and stocks of spent fuel at nuclear facilities.

may continue to hold it, which entails an instantaneous holding cost[3] $h(e) \in H$, or emit it into the public stock, which entails an emission penalty l paid to the regulator. The firm is required to pay the holding cost on a unit of pollutant as long as it stays in the firm's private stock, while the emission penalty is a one-time charge that transfers the responsibility of processing the pollutant from the firm to the regulator.

The holding cost $h(e)$ is chosen by the firm *via* the choice of technology $e \in E = [e_0, e_1]$ at price $\psi(e)$. The emission penalty l represents the cost incurred by the regulator in implementing the socially mandated standard of emission treatment. Given this perspective, l reflects the social desire to mitigate the negative externalities created by the public pollution stock. We take l as exogenously given.

We focus on the trade-off between internal processing and emission by assuming away the possibility of adjusting the activity level.[4] This is done by representing the activity levels by an exogenously given stochastic process determined by parameters x, μ and σ. The activity process determines x, μ and σ^2 as the mean, drift and variance, respectively, of the firm's internal pollution stock process.[5] We amplify this description in Section 2.1.

Given the parameters described above, let $v(x, \mu, \sigma, \alpha)$ be the value of the firm's operations in the private good market, let $V(x, \mu, \sigma)$ be the value of the firm to consumers, and let $C(x, \mu, \sigma, h(e), 0, l, \alpha)$ be the cost to the firm of implementing the optimal policy of managing its private pollution stock. We take x, μ, σ and l as exogenously given and common knowledge, and therefore, suppress them in our expressions. Therefore, firm α's market value prior to regulation is $v(\alpha)$, its value to consumers is V, and the cost of optimally managing its private pollution stock is $c(\alpha, e) = C(\alpha, h(e))$. Other variations of this setting can be formulated by substituting α with x, μ or σ.

In our study of the contracting problem, we refer to A as the firm's type space and

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[3] We use "holding cost" interchangeably with "processing cost".

[4] There are situations in which supply adjustment may be ruled out. For instance, supply might be governed by inflexible long-term contracts. Alternatively, the cost of managing pollution might be insignificant relative to the profitability of the firm's private good, so that the firm might have no reason to adjust its activity level in response to the regulatory contract.

[5] A negative drift represents a natural rate of decay of the pollutant.

$E \times \Re$ as the outcome space. A pair $\langle A, (e,T)\rangle$ is called a direct mechanism with $(e,T): A \to E \times \Re$ as the outcome function. Firm α's utility from outcome $(e,T) \in E \times \Re$ is

$$u(\alpha, e, T) = v(\alpha) - c(\alpha, e) - \psi(e) + T \tag{1.1.1}$$

Given the direct mechanism $\langle A, (e,T)\rangle$, firm α's utility from reporting type α' is $U(\alpha, \alpha') = u(\alpha, e(\alpha'), T(\alpha'))$.[6]

We consider mechanisms that induce participation and self-selection by all types. The individual rationality (IR) and incentive compatibility (IC) constraints that characterize such contracts are: for all α, $\alpha' \in A$

$$U(\alpha, \alpha) \geq 0 \qquad \text{and} \qquad U(\alpha, \alpha) \geq U(\alpha, \alpha') \tag{1.1.2}$$

The regulator's welfare is

$$W(\alpha, e, T) = V + u(\alpha, e, T) - (1+\lambda)T = V + v(\alpha) - c(\alpha, e) - \psi(e) - \lambda T \tag{1.1.3}$$

where $1+\lambda$, with $\lambda \in \Re_{++}$, is the social shadow value *per* unit of payment by the regulator to the firm.

If technology choice e is contractible and α is known to the firm but not known to the regulator, then we seek to design a direct mechanism $(e,T): A \to E \times \Re$, such that (e,T) maximizes the expectation of (1.1.3) subject to (1.1.2).

This framework can be interpreted as the regulator making the firm responsible for processing the private *and* public pollution stock in exchange for a transfer from the regulator to the firm. While the firm is required to process the public pollution in the mandated manner, it is free to choose the internal processing technology. The regulator designs the contract to provide incentives to the firm to adopt the socially desirable technology, which is determined endogenously by the model.

1.2 Firm's cost function

[6] We restrict attention to direct mechanisms without loss of generality because of the revelation principle (see Fudenberg and Tirole, 1991). Henceforth, we refer to outcome functions as contracts.

Of the framework described above, we take V, v and ψ as given. The derivation of the firm's cost function c will take into account the dynamic and stochastic nature of the firm's operational environment with respect to pollution management.

The necessity of a dynamic specification is dictated by our problem. In our model, the firm has to choose between accumulating the pollution it creates and emitting it. This choice is of interest only if the costs of the alternative actions are different in nature. A natural difference suggested by the alternatives themselves is that accumulating the pollution imposes a cost in perpetuity while emitting it implies a one-time cost. Given this cost structure, the firm faces a dynamic problem whose solution at every instant will depend on the past, *via* the inherited private pollution stock, and expectations about the future evolution of the stock. A sensible specification of this problem requires a dynamic environment that generates the actual and expected evolution of the pollution stock.

Of the variables that determine pollution, some are controlled by the firm, such as the choice of technology and the supply decision, and some are not, such as commodity prices, the quality of delivered inputs, and the efficacy of the processing technology. The firm may protect itself against some of these shocks through forward and contingent contracts, but to the extent that markets and contracts are incomplete, the firm must perceive the residual shocks as random events. The stochastic nature of our model is intended to capture this residual uncertainty.

1.3 The literature

Using the terminology of Xepapadeas (1997), this paper is concerned with the problem of regulating "point-source stock pollution", i.e., the pollutant is durable and the emitter's identity and the quantity of emissions are perfectly observable.[7] Our approach to this problem may be described as a multi-directional extension of the dynamic emission choice model (ECM) presented in Sections 1 to 3 of Chapter 3 in Xepapadeas (1997).[8]

[7] The techniques used to analyze the model are standard. Our derivation of the firm's cost function relies on stochastic control theory (see Harrison and Taylor, 1978), while the derivation of the optimal regulatory contract is an application of the theory of optimal mechanism design (see Fudenberg and Tirole, 1991).

[8] The ECM is a representative of many similar models, including those in Brock (1977), d'Arge and Kogiku (1973), Forster (1973), Keeler et al. (1971), Mäler (1974), Plourde (1972), and Xepapadeas (1992).

In the ECM, the public stock of pollution is the state variable and the emission rate is the control variable. Given the control trajectory η and the state trajectory S, the regulator's welfare is $w(\eta, S, \alpha)$, where α is a firm-specific parameter that is common knowledge. The law of motion for the state imposes a dynamic constraint in the form of a differential equation, say $L(\eta, S) = 0$. The regulator's optimal control trajectory $\eta(\alpha)$ maximizes $w(\eta, S, \alpha)$ subject to the constraint $L(\eta, S) = 0$. With appropriate specifications, this is a standard optimal control problem.

Usually, the polluting firms and the regulator are distinct entities, with the emissions trajectory being chosen by the firms and not the regulator. In such situations, can the regulator induce a competitive firm α to choose $\eta(\alpha)$? It is straightforward to show that the regulator can decentralize $\eta(\alpha)$ by imposing a firm-specific time-varying Pigouvian tax $\tau(\alpha)$, whose value at any moment is the social shadow cost of the pollution stock at that moment. This implementation of $\eta(\alpha)$ relies on three substantive features of the ECM: (a) the regulator knows α, (b) the firm's decision problem at each instant is static, and (c) the firm is a price-taker. Our model differs from the ECM in all these aspects.

First, α is not common knowledge in our model. Consequently, it is impossible for the regulator, who does not know α, to implement the socially optimal emissions trajectory *via* appropriate Pigouvian taxes that are conditioned on α.

Second, while the ECM identifies the creation of pollutants with emissions, thereby leading to a static emission problem for the firm, our model specifies a dynamic emission problem for the firm, as outlined in Section 1.2.

1.4 Plan of paper and results

Section 2 of this paper answers the following question: given the firm's operational environment and its pollution processing technology, what is the optimal policy for the firm with respect to the decision whether to emit pollution or to process it internally? This question is answered in Theorem 2.3.19 by constructing a stochastic dynamic programming problem whose solution yields the firm's optimal policy and cost as functions of technological and regulatory parameters.

Section 3 characterizes the optimal regulatory contracts subject to implementability constraints. The cost function derived in Section 2 is used to define the incentive constraints in this section. We consider two contracting problems, one with a finite number of types and the other with a continuum of types. The optimal contract for the former problem is stated in Theorem 3.3.13, while the solution of the latter problem is contained in Theorem 3.4.15.

Section 4 concludes the paper with suggestions for extensions of the work reported in this paper.

2. The firm's cost function

2.1 Formal setting

In this section we introduce the formalism and notation that will be used throughout this paper. $\mathcal{N}$ denotes the set of natural numbers, $\Re$ (resp. $\Re_+$, $\Re_{++}$) the set of real (resp. nonnegative real, positive real) numbers, D and D^2 are the first order and second order differential operators respectively, Δ denotes a jump of a real-valued variable, and $\langle .,. \rangle$ denotes the predictable quadratic variation process (Elliott, 1982, Chapter 10). We make the following assumptions regarding the notation used in Section 1.1.

Assumption 2.1.1. *The following restrictions hold throughout the paper.*

(a) H, E and A are nonempty subsets of $\Re_{++}$,

(b) $x \in \Re_+$; $l \in \Re_{++}$; μ, $\sigma \in \Re$ with $\sigma \neq 0$.

Let Ω be the set of continuous real-valued functions with domain $\Re_+$. The Wiener process $W = (W_t)_{t \in \Re_+}$ is the coordinate process on the stochastic base $(\Omega, \mathcal{F}, (\mathcal{F}_t), Q)$, where $(\mathcal{F}_t)$ is a filtration on Ω, $\sigma(\bigcup_{t \in \Re_+} \mathcal{F}_t) \subset \mathcal{F}$, and Q is the unique (Wiener) measure on $(\Omega, \mathcal{F})$ under which W is a Wiener process with zero drift, unit variance, and starting state 0 Q-a.s. We assume, without loss of generality, that $(\mathcal{F}_t)$ is the right-continuous augmentation of the natural filtration generated by W, and that $\mathcal{F}_0$ includes all the Q-negligible events in $\mathcal{F}$. All processes in this paper are defined with reference to $(\Omega, \mathcal{F}, (\mathcal{F}_t), Q)$.

The firm's production technology is specified by the data $\{f; x, \mu, \sigma\}$. The firm's activity process is $a = (a_t)$; activity level a_t at time t produces goods $f(a_t)$ and causes an uncontrolled variation in the firm's private pollution stock.

The firm's private pollution stock process is $Z = (Z_t)$, with $Z_t = X_t + R_t - L_t$. $X = (X_t)$, with $X_t = x + \mu t + \sigma \ln(a_t)$, is the reference process. The variation in X at time t is interpreted as the uncontrolled variation in Z brought about by activity level choice

a_t. This variation can be positive or negative as fresh production increases the stock but autonomous decay of the stock reduces it; the autonomous decay at time t can have a deterministic as well as a random component. R_t (resp. L_t) represents the total controlled additions to (resp. deductions from) the stock upto time t; variations of Z on account of variations in R and L are called controlled variations. When R and L stay unchanged, variations in Z mimic variations in the reference process X.

Definition 2.1.2. *A control policy is a pair of real-valued processes* $(R, L) = (R_t, L_t)_{t \in \Re_+}$, *defined on the stochastic base* $(\Omega, \mathcal{F}, (\mathcal{F}_t), Q)$, *with sample paths that are non-negative, non-decreasing and right-continuous.*

We specify activity levels exogenously by $a_t = e^{W_t}$. Therefore, the uncontrolled variations in the pollution stock match the variations of $X_t = x + \mu t + \sigma W_t$. X is the Brownian motion on the stochastic base $(\Omega, \mathcal{F}, (\mathcal{F}_t), Q)$ with mean x, drift μ and variance σ^2. The properties of Brownian motion imply that the statistical properties of future variations in X are not affected by the current level of X. Since W is a continuous process, so is X.

Given X, Z may take negative values. This is not a consequence of X being able to take negative values. Even if X is chosen to be a positive process, such as the geometric Brownian motion, Z can become negative if L reaches a positive level. Given our definition of Z, we can ensure that it stays non-negative by manipulating the control policy (R, L) appropriately. We force the firm to choose the controls appropriately by specifying the instantaneous processing cost of the pollution stock x as

$$H(x) = \begin{cases} hx, & \text{if } x \geq 0 \\ \infty, & \text{if } x < 0 \end{cases}$$

where $h > 0$. This implies that a cost-minimizing firm will never want the stock to become negative. In order to avoid a negative pollution stock, two conditions have to be met. First, X has to be a continuous process; otherwise, Z might involuntarily jump down from a positive to a negative level. Secondly, if Z is at 0 and X is falling, the firm should be able to raise R sufficiently to offset the falling X. We satisfy the first requirement by specifying X as the Brownian motion, and show by construction that the second requirement can be satisfied.

The unit cost of instantaneously decreasing (resp. increasing) the stock is l (resp. 0); i.e., the price at which a unit of pollution can be sold to the regulator is $-l$ and the price at which pollution can be bought from the regulator is 0. Consequently, the firm 'buying' pollution is a purely technical ruse without economic interest and it has no effect on the firm's cost function. Technically, this ploy will turn out to be equivalent to the imposition of a reflecting barrier on Z at 0.

R and L are decomposed into continuous and jump parts as follows. Let $T_0 = 0$. Given a stopping time T_n, define $T_{n+1} = \inf\{t > T_n \mid Z_t \neq Z_{t-}\}$; thus, T_n is the random time of the n-th jump in the value of Z. Associate with T_n the random variable $\Delta Z_{T_n} = Z_{T_n} - Z_{T_n -} \in \mathcal{F}_{T_n}$; this describes the size of the jump in the value of Z at T_n. Define $\Delta R_{T_n} = \Delta Z_{T_n} \vee 0$ and $\Delta L_{T_n} = -(\Delta Z_{T_n} \wedge 0)$. ΔR_{T_n} (resp. ΔL_{T_n}) is the size of the upward (resp. downward) jump of Z at T_n; it is equal to 0 if Z jumps downwards (resp. upwards) at T_n. Given $t \in \Re_+$, let $N(t) = \sup\{n \in \mathcal{N} \cup \{0\} \mid T_n \leq t\}$; this random variable counts the number of jumps of Z upto time t. Thus, $\sum_{n=0}^{N(t)} \Delta R_{T_n}$ (resp. $\sum_{n=0}^{N(t)} \Delta L_{T_n}$) is the sum of the upwards (resp. downwards) jumps in Z and R (resp. L) upto time t. These definitions allow us to decompose R and L into continuous and jump parts as follows:

$$R_t = \rho_t + \sum_{n=0}^{N(t)} \Delta R_{T_n} \qquad \text{and} \qquad L_t = \lambda_t + \sum_{n=0}^{N(t)} \Delta L_{T_n}$$

where ρ_t and λ_t are the continuous components of R_t and L_t respectively.

Definition 2.1.3. *A control policy* (R, L) *is said to be feasible if*

(a) $Q\left(\cap_{t \in \Re_+} \{\omega \in \Omega \mid (X_t + R_t - L_t)(\omega) \geq 0\}\right) = 1$,

(b) $Q\left(\{\omega \in \Omega \mid \lim_{n \uparrow \infty} T_n(\omega) = \infty\}\right) = 1$, *and*

(c) $\lim_{t \uparrow \infty} E e^{-\alpha t}(X_t + R_t - L_t) = 0$.

Condition (a) requires that the stock process Z should be nonnegative almost surely; (b) requires that the number of jumps of Z in a bounded period be finite almost surely; (c) is a regularity condition. We note the following facts. As R and L are non-decreasing, they are of bounded variation; therefore, so is $R - L$. As W is a continuous martingale and $R - L$ is of bounded variation, Z is a semimartingale (Elliott, 1982, Chapter 12). Given that all processes in this paper are adapted to $(\mathcal{F}_t)$ and right-continuous, they are progressively measurable (Elliott, 1982, Theorem 2.32).

2.2 Basic lemmas

The following lemma is a consequence of the basic change-of-variable formula of stochastic calculus. It will be used in Section 2.3 to characterize the cost function.

Lemma 2.2.1. *Let* $\Gamma = (\sigma^2/2)D^2 + \mu D - \alpha I$. *If*

$$\Phi \in \mathcal{C}^2(\Re_+, \Re) \quad \textit{and} \quad E\int_{(0,t]} ds\, e^{-2\alpha s}[D\Phi(Z_s)]^2 < \infty$$

then

$$Ee^{-\alpha t}\Phi(Z_t) = \Phi(x) + E\left[\int_{(0,t]} e^{-\alpha s}D\Phi(Z_s)d(\rho-\lambda)_s + \int_{(0,t]} ds\, e^{-\alpha s}\Gamma\Phi(Z_s) + \sum_{n=0}^{N(t)} e^{-\alpha T_n}\Delta\Phi(Z)_{T_n}\right]$$

The cost of implementing a control policy (R, L) over the period $[0, t]$ is

$$C_t(R, L; x, \mu, \sigma, h, 0, l, \alpha) = h\int_{[0,t]} ds\, e^{-\alpha s}(X_s + R_s - L_s) + l\int_{[0,t]} e^{-\alpha s}dL_s$$

The cost of implementing (R, L) over $\Re_+$ is

$$C_\infty(R, L; x, \mu, \sigma, h, 0, l, \alpha) = \limsup_{t\uparrow\infty} C_t(R, L; x, \mu, \sigma, h, 0, l, \alpha)$$

Lemma 2.2.2. *Given the parameters* $(x, \mu, \sigma, h, 0, l, \alpha)$,

$$EC_\infty(R, L; x, \mu, \sigma, h, 0, l, \alpha) = \frac{hx}{\alpha} + \frac{h\mu}{\alpha^2} + EC_\infty(R, L; x, \mu, \sigma, 0, h/\alpha, l - h/\alpha, \alpha)$$

This means the problem of specifying (R, L) to minimize $EC_\infty(R, L; x, \mu, \sigma, h, 0, l, \alpha)$ is equivalent to the problem of specifying (R, L) to minimize $EC_\infty(R, L; x, \mu, \sigma, 0, h/\alpha, l - h/\alpha, \alpha)$. Consider an alternative to the given method of costing: when a unit enters the stock, it is charged its infinite horizon processing cost h/α and when it leaves the stock, it is charged the unit cost of leaving, l, less the implicit saving in the processing cost h/α. Lemma 2.2.2 implies that the two methods of costing are identical, *modulo* a constant.

Definition 2.2.3. *A feasible control policy* (R, L) *is said to be optimal if*

$$EC_\infty(R, L; x, \mu, \sigma, h, 0, l, \alpha) \le EC_\infty(R', L'; x, \mu, \sigma, h, 0, l, \alpha)$$

for every feasible control policy (R', L'). *If* (R, L) *is an optimal control policy, let*

$$C(x, \mu, \sigma, h, 0, l, \alpha) = E_0C_\infty(R, L; x, \mu, \sigma, h, 0, l, \alpha)$$

2.3 Optimal policy and the cost function

Lemma 2.3.1 characterizes a lower bound on the cost of implementing a feasible control policy. We shall go on to construct a control policy whose cost attains this lower bound, implying that it is an optimal policy. Theorem 2.3.19 states the optimal control policy and the resulting cost function.

Lemma 2.3.1. *(Optimality criterion) Suppose $\Phi : \Re_+ \to \Re$ is such that*

(a) $\Phi \in \mathcal{C}^2(\Re_+, \Re)$,

(b) $-\delta \le D\Phi \le l - \delta$, and

(c) $\Gamma\Phi \ge 0$.

If (R, L) is a feasible control policy, then $\Phi(x) \le EC_\infty(R, L; x, \mu, \sigma, 0, \delta, l - \delta, \alpha)$ for every $x \in \Re_+$.

Given $S > 0$, let $f(.; S) : \Re_+ \to \Re$ be such that

$$Df(0; S) = -\delta \qquad Df(S; S) = l - \delta \qquad \text{and} \qquad \Gamma f(x; S) = 0 \tag{2.3.2}$$

for every $x \in (0, S)$. Given $S \in \Re_{++}$, and $f(.; S)$ that solves (2.3.2), define $F(.; S) : \Re_+ \to \Re$ by

$$F(x; S) = \begin{cases} f(x; S), & \text{if } x \in [0, S] \\ f(S; S) + (l - \delta)(x - S), & \text{if } x > S \end{cases} \tag{2.3.3}$$

The roots of the characteristic polynomial of $\Gamma f(.; S)$ are $\lambda_1 = -\beta - \gamma < 0$ and $\lambda_2 = -\beta + \gamma > 0$, where

$$\beta = \mu/\sigma^2 \qquad \text{and} \qquad \gamma = (\beta^2 + 2\alpha/\sigma^2)^{1/2}$$

Given $S > 0$, the unique solution of (2.3.2) is

$$f(x; S) = ae^{\lambda_1 x} + be^{\lambda_2 x} \tag{2.3.4}$$

where

$$a = \frac{\delta e^{\gamma S} + (l - \delta)e^{\beta S}}{(\beta + \gamma)(e^{\gamma S} - e^{-\gamma S})} \qquad \text{and} \qquad b = \frac{\delta e^{-\gamma S} + (l - \delta)e^{\beta S}}{(\gamma - \beta)(e^{\gamma S} - e^{-\gamma S})}$$

It is clear from (2.3.2) and (2.3.3) that $F(.; S)$ and $DF(.; S)$ are continuous. Clearly,

$$D^2F(x; S) = \begin{cases} D^2 f(x; S), & \text{if } x \in (0, S) \\ 0, & \text{if } x \in (S, \infty) \end{cases}$$

Clearly, $D^2F(.; S)$ is continuous at S if and only if $\lim_{x \uparrow S} D^2F(x; S) = 0$. Routine calculations reveal that this condition is satisfied if and only if $S > 0$ solves the following equation:

$$(l - \delta)\left[(\gamma + \beta)e^{(\beta - \gamma)S} + (\gamma - \beta)e^{(\beta + \gamma)S}\right] = -2\gamma\delta \tag{2.3.5}$$

Lemma 2.3.6. *If $\delta > l$, then there exists a unique $S > 0$ that solves (2.3.5). (2.3.5) has no solution if $\delta \le l$.*

The next result shows that F defined by (2.3.3) satisfies assumptions (a), (b) and (c) of Lemma 2.3.1.

Lemma 2.3.7. *Suppose $\delta > l$, S is the unique solution of (2.3.5), $f(.;S)$ is defined by (2.3.4), and $F(.;S)$ is defined by (2.3.3). Then,*

(A) $F(.;S) \in \mathcal{C}^2(\Re_+, \Re)$,

(B) $-\delta \leq DF(.;S) \leq l - \delta$, and

(C) $\Gamma F(.;S) \geq 0$.

Combining Lemmas 2.3.1 and 2.3.7, we immediately have

Lemma 2.3.8. *Suppose $\delta > l$, S is the unique solution of (2.3.5), $f(.;S)$ is defined by (2.3.4), and $F(.;S)$ is defined by (2.3.3). Then,*

$$F(x;S) \leq EC_\infty(R, L; x, \mu, \sigma, 0, \delta, l - \delta, \alpha) \tag{2.3.9}$$

for every $x \in \Re_+$ and feasible control policy (R, L).

We now construct a feasible control policy (R, L) such that equality holds in (2.3.9). By Lemma 2.3.8, this policy will be an optimal control policy.

Let $S > 0$ be given by (2.3.5). Consider the equations: for every $t \in \Re_+$,

$$R_t = \sup\left\{[L_u - X_u]^+ \mid u \in [0, t]\right\} \tag{2.3.10a}$$

and

$$L_t = \sup\left\{[X_u + R_u - S]^+ \mid u \in [0, t]\right\} \tag{2.3.10b}$$

The policy implied by (2.3.10) amounts to imposing a lower reflecting barrier on Z at 0 and an upper reflecting barrier on Z at S. Informally, L (resp. R) grows only at random times when Z hits the upper (resp. lower) barrier S (resp. 0) and X is rising (resp. falling), with the rise in L (resp. R) being just sufficient to exactly offset the growth (resp. decay) of X.

Lemma 2.3.11. *Suppose $S > 0$.*

(A) If (R, L) is a pair of processes that satisfy (2.3.10) for every $t \in \Re_+$, then (R, L) is a continuous feasible control policy with $R_0 = 0$ and $L_0 = [X_0 - S]^+$.

(B) There exists a unique solution of (2.3.10), say (R, L).

(C) Suppose $\delta > l$, $S > 0$ is the unique solution of (2.3.5), f is defined by (2.3.2), F is defined by (2.3.3), and (R, L) is the unique solution of (2.3.10) given S. Then, for every $x \in \Re_+$,

$$EC_\infty(R, L; x, \mu, \sigma, 0, \delta, l - \delta, \alpha) = F(x)$$

Lemmas 2.3.8 and 2.3.11(C) yield the firm's cost function when $\delta > l$. We now turn to the problem when $\delta \le l$.

Let $\phi : \Re \to \Re$ be such that

$$D\phi(0) = -\delta \qquad \lim_{x\uparrow\infty} \phi(x) = 0 \qquad \lim_{x\uparrow\infty} D\phi(x) = 0 \qquad \text{and} \qquad \Gamma\phi(x) = 0 \tag{2.3.12}$$

for every $x \in \Re_+$. It is easy to check that the unique solution of (2.3.12) is

$$\phi(x) = \frac{\delta}{\beta+\gamma} e^{-(\beta+\gamma)x} \tag{2.3.13}$$

The following lemma notes that ϕ satisfies the assumptions of Lemma 2.3.1.

Lemma 2.3.14. *Suppose $\delta \le l$ and ϕ is defined by (2.3.13). Then,*

(A) $\phi \in \mathcal{C}^2(\Re_+, \Re)$,

(B) $-\delta \le D\phi \le l - \delta$, and

(C) $\Gamma\phi \ge 0$.

Combining Lemmas 2.3.1 and 2.3.14 yields

Lemma 2.3.15. *Suppose $\delta \le l$ and ϕ is defined by (2.3.13). Then,*

$$\phi(x) \le E_0 C_\infty(R, L; x, \mu, \sigma, 0, \delta, l-\delta, \alpha) \tag{2.3.16}$$

for every $x \in \Re_+$ and feasible control policy (R, L).

We now construct a feasible policy (R, L) such that equality holds in (2.3.16). This policy will be the optimal policy when $\delta \le l$. Consider the equations

$$R_t = \sup\left\{[-X_u]^+ \mid u \in [0,t]\right\} \qquad \text{and} \qquad L_t = 0, \qquad t \in \Re_+ \tag{2.3.17}$$

The policy implied by (2.3.17) amounts to imposing a lower reflecting barrier on Z at 0 and no upper barrier. The proof of the following result mimics that of Lemma 2.3.11.

Lemma 2.3.18. *(A) If (R, L) is a pair of processes that satisfy (2.3.17) for every $t \in \Re_+$, then (R, L) is a continuous feasible control policy with $(R_0, L_0) = (0, 0)$.*

(B) There exists a unique solution of (2.3.17), say (R, L).

(C) Suppose $\delta \le l$, ϕ is defined by (2.3.14) and (R, L) is the unique solution of (2.3.17). Then, for every $x \in \Re_+$,

$$EC_\infty(R, L; x, \mu, \sigma, 0, \delta, l-\delta, \alpha) = \phi(x)$$

Lemmas 2.3.15 and 2.3.18(C) yield the firm's cost function when $\delta \le l$. Combining Lemmas 2.2.2, 2.3.11 and 2.3.18, we have

Theorem 2.3.19. *Given parameters $(x, \mu, \sigma, h, 0, l, \alpha)$, the unique optimal control policy when $l < h/\alpha$ is given by (2.3.10) and the unique optimal control policy when $l \geq h/\alpha$ is given by (2.3.17), with cost function*

$$C(x, \mu, \sigma, h, 0, l, \alpha) = \frac{hx}{\alpha} + \frac{h\mu}{\alpha^2} + \begin{cases} F(x), & \text{if } h > l\alpha \\ \phi(x), & \text{if } h \leq l\alpha \end{cases} \tag{2.3.20}$$

We conclude this section by noting some simple properties of the optimal control policy and the corresponding cost function. First, the *feasibility* of a control policy (R, L) is independent of the parameters (h, l, α); it is entirely dependent on the parameters (x, μ, σ). Secondly, the *nature* of the optimal policy, i.e., whether the pollution stock is bounded above by a reflecting barrier at some finite S or not, is determined entirely by the parameters $(h, 0, l, \alpha)$.

Let $C(x, \mu, \sigma, h, 0, l, \alpha) = EC_\infty(R, L; x, \mu, \sigma, h, 0, l, \alpha)$. If $\alpha' > \alpha > 0$, then (R, L) continues to be a feasible control policy. Consequently,

$$\begin{aligned} C(x, \mu, \sigma, h, 0, l, \alpha) &= EC_\infty(R, L; x, \mu, \sigma, h, 0, l, \alpha) \\ &\geq EC_\infty(R, L; x, \mu, \sigma, h, 0, l, \alpha') \\ &\geq C(x, \mu, \sigma, h, 0, l, \alpha') \end{aligned}$$

i.e., C is decreasing in α. By analogous arguments, it follows that C is increasing in h and l. Given $t > 0$, let $C(x, \mu, \sigma, th, 0, tl, \alpha) = EC_\infty(R, L; x, \mu, \sigma, th, 0, tl, \alpha)$. As $C_\infty(R, L; x, \mu, \sigma, th, 0, tl, \alpha) = tC_\infty(R, L; x, \mu, \sigma, h, 0, l, \alpha)$, we have

$$C(x, \mu, \sigma, th, 0, tl, \alpha) = tEC_\infty(R, L; x, \mu, \sigma, h, 0, l, \alpha) \geq tC(x, \mu, \sigma, h, 0, l, \alpha) \tag{2.3.21}$$

By a similar argument, C is super-additive in (h, l):

$$C(x, \mu, \sigma, h + h', 0, l + l', \alpha) \geq C(x, \mu, \sigma, h, 0, l, \alpha) + C(x, \mu, \sigma, h', 0, l', \alpha) \tag{2.3.22}$$

Combining (2.3.21) and (2.3.22), we see that C is concave in (h, l).

3. Regulation problem

3.1 Formulation

We employ the notation and formalism outlined in Section 1.1. The firm's value prior to environmental regulation is given by $v : A \to \Re$. Let $h : E \to H$ yield the processing cost as a function of technology. Define $c : A \times E \to \Re$ by $c(\alpha, e) = C(\alpha, h(e))$. Let $\psi : E \to \Re$ generate the price paid by the firm for technology.

Assumption 3.1.2. *h and ψ are twice continuously differentiable on the interiors of their domains with*

(a) $Dh < 0$ and $D^2h > 0$,

(b) $D\psi > 0$ and $D^2\psi > 0$,

(c) $\sup_{e\in E} u(\alpha, e, 0) > 0$ for every $\alpha \in A$,

(d) $D_{122}c < 0$,

(e) $Dv - D_1c > 0$, and

(f) $D_{22}c + D^2\psi > 0$.

(a) means that the unit processing cost decreases with greater technology but technology faces diminishing returns in terms of cost savings. (b) implies that the cost of technology acquisition increases at an increasing rate with technology. (c) and (e) imply the satisfaction of second order conditions.

3.2 Preliminaries

The firm's utility function $u : A \times E \times \Re \to \Re$ is given by (1.1.1) and the regulator's welfare function $W : A \times E \times \Re \to \Re$ is given by (1.1.3). The IR and IC constraints that induce participation and self-selection by all types are given by (1.1.2). If truth-telling is incentive compatible, then the regulator's welfare can be written as

$$W(\alpha, e(\alpha), T(\alpha)) = V + u(\alpha, e(\alpha), 0) - \lambda T(\alpha) = V + (1+\lambda)u(\alpha, e(\alpha), 0) - \lambda U(\alpha, \alpha) \quad (3.2.1)$$

The following is a characterization of contracts that satisfy the incentive compatibility conditions for all types.

Lemma 3.2.2. *Given contract (e, T) and $D_{12}c > 0$ (resp. $D_{12}c < 0$),*

$$U(\alpha, \alpha) \geq U(\alpha, \alpha'), \quad \forall \alpha \in A, \forall \alpha' \in A$$

iff. $D_2U(\alpha, \alpha) = 0$ for almost every $\alpha \in A$ and e is non-increasing (resp. non-decreasing).

We now consider two special regulatory regimes. First, consider the regime in which the firm pays l for each emitted unit of pollution without a transfer. Consequently, firm α selects technology $e \in E$ to maximize $u(\alpha, e, 0)$ subject to the constraint $u(\alpha, e, 0) \geq 0$.

Assuming the optimal choice $e(\alpha) \in (e_0, e_1)$ and $u(\alpha, e(\alpha), 0) > 0$, the following first order condition must hold:

$$D_2 c(\alpha, e(\alpha)) + D\psi(e(\alpha)) = 0 \tag{3.2.3}$$

Assumption 3.1.2 implies the second order condition. Firm α's utility is $u(\alpha, e(\alpha), 0)$ and the regulator's welfare is $W(\alpha, e(\alpha), 0) = V + u(\alpha, e(\alpha), 0)$. It follows from (3.2.3) that

$$De(\alpha) = \frac{-D_{12}c(\alpha, e(\alpha))}{D_{22}c(\alpha, e(\alpha)) + D^2\psi(e(\alpha))}$$

Therefore, if $D_{12}c > 0$ (resp. $D_{12}c < 0$), then $De < 0$ (resp. $De > 0$).

Alternatively, suppose the regulator has complete information. Given α, the regulator chooses a constant contract (e, T) to maximize $W(\alpha, e, T)$ subject to the individual rationality constraint $u(\alpha, e, T) \geq 0$. The optimal contract for type α, say (e_α, T_α), is characterized by the conditions

$$T_\alpha = \psi(e_\alpha) + c(\alpha, e_\alpha) - v(\alpha) \qquad \text{and} \qquad D_2 c(\alpha, e_\alpha) + D\psi(e_\alpha) = 0 \tag{3.2.4}$$

The second condition is identical to (3.2.3), which implies that $e_\alpha = e(\alpha)$. The first condition amounts to setting

$$U(\alpha, \alpha) = u(\alpha, e_\alpha, T_\alpha) = u(\alpha, e(\alpha), T_\alpha) = u(\alpha, e(\alpha), 0) + T_\alpha = 0$$

which implies $T_\alpha = -u(\alpha, e(\alpha), 0) < 0$. The regulator's welfare is

$$W(\alpha, e_\alpha, T_\alpha) = V + (1 + \lambda)u(\alpha, e(\alpha), 0) > W(\alpha, e(\alpha), 0)$$

as $u(\alpha, e(\alpha), 0) > 0$. Consequently, social welfare is higher under the second regime. The transfer from the firm to the regulator serves to eliminate the entire rent that accrued to the firm in the first regime.

3.3 Contracting under incomplete information: discrete case

Assumption 3.3.1. *$\underline{\alpha}$, $\overline{\alpha} \in A$ such that $\underline{\alpha} < \overline{\alpha}$, and F is a distribution function on A such that* $\operatorname{supp} F = \{\underline{\alpha}, \overline{\alpha}\}$ *and $F(\underline{\alpha}) = p \in (0, 1)$.*

In this section, the firm's type is $\alpha \in \{\underline{\alpha}, \overline{\alpha}\}$, which is private information, and F is the regulator's belief about α, which is common knowledge. Consider an equilibrium in which the regulator offers a contract $\{(\underline{e}, \underline{T}), (\overline{e}, \overline{T})\}$, firm $\underline{\alpha}$ chooses $(\underline{e}, \underline{T})$ and firm $\overline{\alpha}$ chooses $(\overline{e}, \overline{T})$. In such an equilibrium, the following conditions must hold:

$$u(\underline{\alpha}, \underline{e}, \underline{T}) \geq 0 \tag{3.3.2}$$

$$u(\overline{\alpha}, \overline{e}, \overline{T}) \geq 0 \tag{3.3.3}$$

$$u(\underline{\alpha}, \underline{e}, \underline{T}) \geq u(\underline{\alpha}, \overline{e}, \overline{T}) \tag{3.3.4}$$

$$u(\overline{\alpha}, \overline{e}, \overline{T}) \geq u(\overline{\alpha}, \underline{e}, \underline{T}) \tag{3.3.5}$$

(3.3.2) and (3.3.3) are the IR constraints for the two types, and (3.3.4) and (3.3.5) are their IC constraints. Define $\Phi : E \to \Re$ by

$$\Phi(e) = u(\overline{\alpha}, e, 0) - u(\underline{\alpha}, e, 0) = \int_{\underline{\alpha}}^{\overline{\alpha}} d\alpha \, [Dv(\alpha) - D_1 c(\alpha, e)]$$

It follows that $D\Phi = -\int_{\underline{\alpha}}^{\overline{\alpha}} d\alpha \, D_{12}c(\alpha, .)$ and $D^2\Phi = -\int_{\underline{\alpha}}^{\overline{\alpha}} d\alpha \, D_{122}c(\alpha, .)$. Assumption 3.1.2 implies that $\Phi > 0$ and $D^2\Phi > 0$. $\Phi(e)$ is the rent earned by type $\overline{\alpha}$ from technology e. Given $\underline{U} = u(\underline{\alpha}, \underline{e}, \underline{T})$ and $\overline{U} = u(\overline{\alpha}, \overline{e}, \overline{T})$, constraints (3.3.2) to (3.3.5) can be re-written as

$$\underline{U} \geq 0 \tag{3.3.6}$$

$$\overline{U} \geq 0 \tag{3.3.7}$$

$$\underline{U} \geq \overline{U} - \Phi(\overline{e}) \tag{3.3.8}$$

$$\overline{U} \geq \underline{U} + \Phi(\underline{e}) \tag{3.3.9}$$

The regulator's optimal contract $\{(\underline{U}, \underline{e}), (\overline{U}, \overline{e})\}$ maximizes

$$p\,[(1+\lambda)u(\underline{\alpha}, \underline{e}, 0) - \lambda \underline{U}] + (1-p)\left[(1+\lambda)u(\overline{\alpha}, \overline{e}, 0) - \lambda \overline{U}\right] \tag{3.3.10}$$

subject to constraints (3.3.6) to (3.3.9).

(3.3.6) and (3.3.9) imply $\overline{U} \geq \underline{U} + \Phi(\underline{e}) \geq \Phi(\underline{e}) \geq 0$. Thus, (3.3.7) is satisfied if (3.3.6) and (3.3.9) are satisfied. If (3.3.7) is binding, then $0 \geq \underline{U} + \Phi(\underline{e})$, i.e., $\underline{U} \leq -\Phi(\underline{e}) < 0$, which violates (3.3.6). Thus, $\overline{U} > 0$. If $\underline{U} > 0$ at the optimum, then both $\underline{U}$ and $\overline{U}$ can be reduced by $\epsilon > 0$, sufficiently small, without violating constraints (3.3.6) to (3.3.9). As this increases the value of the objective, we have a contradiction. Consequently, at the optimum, we must have $\underline{U} = 0$ and $\Phi(\underline{e}) \le \overline{U} \le \Phi(\overline{e})$. Clearly, at the optimum, we must have $\overline{U} = \Phi(\underline{e})$. This simplifies the regulator's problem to: choose $\underline{e}$ and $\overline{e}$ to maximize

$$p(1+\lambda)u(\underline{\alpha}, \underline{e}, 0) + (1-p)\left[(1+\lambda)u(\overline{\alpha}, \overline{e}, 0) - \lambda\Phi(\underline{e})\right]$$

The first-order conditions characterizing the optimal choices are

$$D_2c(\overline{\alpha}, \overline{e}) + D\psi(\overline{e}) = 0 \tag{3.3.11}$$

and

$$D_2c(\underline{\alpha}, \underline{e}) + D\psi(\underline{e}) = -\frac{1-p}{p}\frac{\lambda}{1+\lambda}D\Phi(\underline{e}) \tag{3.3.12}$$

Suppose $D_{12}c > 0$. Then, $D\Phi < 0$ and we have

$$D_2c(\overline{\alpha}, \underline{e}) + D\psi(\underline{e}) > D_2c(\underline{\alpha}, \underline{e}) + D\psi(\underline{e}) > 0 = D_2c(\overline{\alpha}, \overline{e}) + D\psi(\overline{e})$$

Since $D_{22}c(\overline{\alpha}, .) + D^2\psi(.) > 0$, we have $\underline{e} > \overline{e}$. Comparing (3.3.11) and (3.3.12) with (3.2.3), we have $\overline{e} = e_{\overline{\alpha}}$ and $\underline{e} > e_{\underline{\alpha}}$, i.e., relative to the full information choices, type $\overline{\alpha}$'s choice is not distorted while type $\underline{\alpha}$'s choice is distorted upwards. We collect these facts in the following result.

Theorem 3.3.13. *Given $D_{12}c > 0$ (resp. $D_{12}c < 0$), and Assumptions 3.1.2 and 3.3.1, $\{(\underline{U}, \underline{e}), (\overline{U}, \overline{e})\}$ is the optimal contract, where $\overline{e}$ and $\underline{e}$ are characterized by (3.3.11) and (3.3.12) respectively, $\overline{U} = \Phi(\underline{e})$ and $\underline{U} = 0$. The implied transfers are $\overline{T} = \Phi(\underline{e}) + c(\overline{\alpha}, \overline{e}) + \psi(\overline{e}) - v(\overline{\alpha})$ and $\underline{T} = c(\underline{\alpha}, \underline{e}) + \psi(\underline{e}) - v(\underline{\alpha})$. Moreover, $\overline{e} = e_{\overline{\alpha}} < e_{\underline{\alpha}} < \underline{e}$ (resp. $\overline{e} = e_{\overline{\alpha}} > e_{\underline{\alpha}} > \underline{e}$).*

3.4 Contracting under incomplete information: continuum case

Assumption 3.4.1. *F is a distribution function on A with a positive density f, and $DG(\alpha) < 0$ where $G(\alpha) = [1 - F(\alpha)]/f(\alpha)$.*

$DG(\alpha) < 0$ is equivalent to $f(\alpha)/[1 - F(\alpha)]$ increasing in α. Many familiar distributions satisfy this monotone hazard rate condition (Bagnoli and Bergstrom, 1989). Suppose the firm's type is $\alpha \in A$, which is private information, and F is the regulator's belief about α, which is common knowledge. We wish to characterize an equilibrium in which the regulator offers a contract (e, T) and firm $\alpha \in A$ chooses to participate and self-selects by choosing $(e(\alpha), T(\alpha))$. Suppose $D_{12}c > 0$.

If (e, T) induces self-selection, then the regulator's welfare from type α is given by (3.2.1). Using (3.2.1) and setting $\mathcal{U}(\alpha) = U(\alpha, \alpha)$, the regulator's expected welfare is

$$\int_{\alpha_0}^{\alpha_1} d\alpha \, f(\alpha) \left[V - (1+\lambda) \left(c(\alpha, e(\alpha)) + \psi \circ e(\alpha) - v(\alpha)\right) - \lambda \mathcal{U}(\alpha)\right] \tag{3.4.2}$$

Therefore, we can change variables and consider contracts $(e, \mathcal{U})$ instead of (e, T). The IR constraints can be written as: for every $\alpha \in A$

$$\mathcal{U}(\alpha) \geq 0 \tag{3.4.3}$$

By Lemma 3.2.2, truth-telling is incentive compatible if and only if, for almost every $\alpha \in A$,

$$D_2 U(\alpha, \alpha) = 0 \tag{3.4.4}$$

and

$$De(\alpha) \leq 0 \tag{3.4.5}$$

(3.4.4) amounts to

$$D\mathcal{U}(\alpha) = D_1 U(\alpha, \alpha) = D_1 u(\alpha, e(\alpha), T(\alpha)) = Dv(\alpha) - D_1 c(\alpha, e(\alpha)) \tag{3.4.6}$$

Condition (3.4.5) can be written as

$$De(\alpha) = -y(\alpha) \qquad \text{and} \qquad y(\alpha) \geq 0 \tag{3.4.7}$$

Assumption 3.1.4 and (3.4.6) imply that $D\mathcal{U}(\alpha) > 0$. Consequently, the IR constraint (3.4.3) can be replaced by

$$\mathcal{U}(\alpha_0) = 0 \tag{3.4.8}$$

It follows that the optimal contract $(e, \mathcal{U})$ will solve the Lagrange optimal control problem of maximizing (3.4.2) subject to (3.4.6), (3.4.7) and (3.4.8).

We shall treat $\mathcal{U}$ and e as state variables, with costate variables μ and ξ respectively, and $y \in \Re_+$ as the control variable. Suppose $(y, \mathcal{U}, e)$ solves the optimal control problem with $e(A) \subset \operatorname{Int} E$. Define the Hamiltonian function

$$\begin{aligned} &H(\alpha, \mathcal{U}, e, \mu, \xi, y) \\ &\qquad = f(\alpha) \left[V - (1+\lambda)\big(c(\alpha, e) + \psi(e) - v(\alpha)\big) - \lambda \mathcal{U}\right] + \mu[Dv(\alpha) - D_1 c(\alpha, e)] - \xi y \end{aligned}$$

Applying Pontryagin's theorem (Cesari, 1983, Theorem 5.1.i), there exists an absolutely continuous function $(\mu, \xi) : A \to \Re^2$ such that

$$D\mu(\alpha) = -D_2 H(\alpha, \mathcal{U}(\alpha), e(\alpha), \mu(\alpha), \xi(\alpha), y(\alpha)) = \lambda f(\alpha) \tag{3.4.9}$$

and

$$\begin{aligned} D\xi(\alpha) &= -D_3 H(\alpha, \mathcal{U}(\alpha), e(\alpha), \mu(\alpha), \xi(\alpha), y(\alpha)) \\ &= f(\alpha)(1+\lambda)\left[D_2 c(\alpha, e(\alpha)) + D\psi \circ e(\alpha)\right] + \mu(\alpha) D_{12} c(\alpha, e(\alpha)) \end{aligned} \tag{3.4.10}$$

for almost all $\alpha \in A$. Since $\mathcal{U}(\alpha_1)$, $e(\alpha_0)$ and $e(\alpha_1)$ are unrestricted, we have the transversality conditions

$$\mu(\alpha_1) = 0 \qquad \xi(\alpha_0) = 0 \qquad \xi(\alpha_1) = 0 \tag{3.4.11}$$

Combining (3.4.9) and (3.4.11) we have

$$-\mu(\alpha) = \mu(\alpha_1) - \mu(\alpha) = \int_\alpha^{\alpha_1} dx\, D\mu(x) = \lambda \int_\alpha^{\alpha_1} dx\, f(x) = \lambda[1 - F(\alpha)] \tag{3.4.12}$$

Furthermore, $y(\alpha)$ maximizes $H(\alpha, \mathcal{U}(\alpha), e(\alpha), \mu(\alpha), \xi(\alpha), y)$ subject to the constraint $y \geq 0$ for almost all $\alpha \in A$.

Regime 1.

Let $\alpha \in \mathrm{Int}\,(A - y^{-1}(0))$, i.e., $y(\alpha) > 0$. It follows that there exists $\epsilon > 0$ such that $y(\alpha') > 0$ for every $\alpha' \in (\alpha - \epsilon, \alpha + \epsilon)$. Then $\xi(\alpha') = 0$ for every $\alpha' \in (\alpha - \epsilon, \alpha + \epsilon)$, i.e., $D\xi(\alpha) = 0$. It follows from (3.4.10) and (3.4.12) that

$$D_2 c(\alpha, e(\alpha)) + D\psi \circ e(\alpha) = \frac{\lambda}{1+\lambda} G(\alpha) D_{12} c(\alpha, e(\alpha)) \tag{3.4.13}$$

(3.4.13) implicitly defines $e(\alpha)$ and (3.4.6) determines $\mathcal{U}(\alpha)$. It follows from (3.4.13) that

$$De(\alpha) = \frac{\lambda G(\alpha) D_{112} c(\alpha, e(\alpha)) + \left[\lambda DG(\alpha) - (1+\lambda)\right] D_{12} c(\alpha, e(\alpha))}{(1+\lambda)[D_{22} c(\alpha, e(\alpha)) + D^2 \psi \circ e(\alpha)] - \lambda G(\alpha) D_{122} c(\alpha, e(\alpha))} = -y(\alpha) < 0$$

To sum up, in Regime 1, $\mathcal{U}$ is non-negative and increasing, μ is negative and increasing, e is positive and decreasing, and ξ is zero.

Regime 2.

Let $\alpha \in y^{-1}(0)$, i.e., $De(\alpha) = -y(\alpha) = 0$. Let

$$\alpha' = \sup(A - y^{-1}(0)) \cap [\alpha_0, \alpha] \qquad \text{and} \qquad \alpha'' = \inf(A - y^{-1}(0)) \cap [\alpha, \alpha_1]$$

By definition, there exists a sequence (α_n) in $(A - y^{-1}(0)) \cap [\alpha_0, \alpha]$ such that $\lim_{n\uparrow\infty} \alpha_n = \alpha'$. By definition, $y(\alpha_n) > 0$ for every $n \in \mathcal{N}$. It follows that $\xi(\alpha_n) = 0$ for every $n \in \mathcal{N}$. By the continuity of ξ, $\xi(\alpha') = \lim_{n\uparrow\infty} \xi(\alpha_n) = 0$. Similarly, $\xi(\alpha'') = 0$. As $y(\alpha) = De(\alpha) = 0$ for every $\alpha \in (\alpha', \alpha'')$, we have $e(\alpha) = \kappa > 0$ for every $\alpha \in (\alpha', \alpha'')$. Since $y = 0$ maximizes $H(\alpha, \mathcal{U}(\alpha), e(\alpha), \mu(\alpha), \xi(\alpha), y)$, it must also maximize $-\xi(\alpha)y$; consequently, we must have $\xi(\alpha) \geq 0$. (3.4.10) yields

$$\xi(\alpha) = (1+\lambda) \int_{\alpha'}^{\alpha} dx\, f(x) \left[D_2 c(x,\kappa) + D\psi(\kappa) - \frac{\lambda}{1+\lambda} G(x) D_{12} c(x,\kappa) \right]$$

such that $\xi(\alpha'') = 0$, i.e.,

$$\int_{\alpha'}^{\alpha''} dx\, f(x) \left[D_2 c(x,\kappa) + D\psi(\kappa) - \frac{\lambda}{1+\lambda} G(x) D_{12} c(x,\kappa) \right] = 0 \qquad (3.4.14)$$

To sum up, in Regime 2, $\mathcal{U}$ is non-negative and increasing, μ is negative and increasing, e is a positive constant κ, and ξ is non-negative.

Theorem 3.4.15. *Suppose $D_{12}c > 0$ (resp. $D_{12}c > 0$), Assumptions 3.1.2 and 3.4.1 hold, and $\{\mathcal{U}, e\}$ is an optimal contract such that $e(A) \subset (e_0, e_1)$. Then,*

(A) $\mathcal{U}$ is increasing on A with $\mathcal{U}(\alpha_0) = 0$; it is determined by (3.4.6),

(B) e is non-increasing (resp. non-decreasing) on A; it is determined by (3.4.13) if $De(\alpha) < 0$ (resp. $De(\alpha) < 0$), and by (3.4.14) if $De(\alpha) = 0$,

(C) μ is negative, increasing, and given by (3.4.12), and

(D) ξ is non-negative (resp. non-positive); it is positive (resp. negative) only if $De(\alpha) = 0$.

4. Extensions

First, the above model can be extended to a situation where one regulator faces many firms. Even if the firms do not interact directly, they will be connected *via* the regulator's budget constraint and the fact that all the firms' emissions add to the same public stock of pollution.

A second extension is to endogenize the mandated clean-up technology for public pollution by choosing l to maximize social welfare.

A third direction for exploration is a numerical analysis of the solution proposed in this paper.

Appendix

Proof of Lemma 2.2.1. By the change of variable formula (Elliott, 1982, Theorem 12.21),

$$\Phi(Z_t) = \Phi(Z_0) + \int_{(0,t]} D\Phi(Z_{s-})dZ_s + \frac{1}{2}\int_{(0,t]} D^2\Phi(Z_{s-})d\langle Z^c, Z^c\rangle_s + \sum_{n=1}^{N(t)}[\Delta\Phi(Z)_{T_n} - D\Phi(Z_{T_n-})\Delta Z_{T_n}]$$

where (Z_t^c) is the continuous martingale part of (Z_t). For $s > 0$, we have

$$Z_s = X_s + \rho_s - \lambda_s + \Delta Z_{T_0} + \sum_{n=1}^{N(s)} \Delta Z_{T_n}$$

where the last term can be re-written as

$$\sum_{n=1}^{N(s)} \Delta Z_{T_n} = \sum_{n=1}^{\infty} \Delta Z_{T_n} 1_{(0,s]}(T_n) = \sum_{n=1}^{\infty} \Delta Z_{T_n} \int_{(0,s]} \delta_{T_n}(du)$$

δ_{T_n} is the Dirac measure sitting at T_n. Analogously,

$$\sum_{n=1}^{N(s)} \Delta\Phi(Z)_{T_n} = \sum_{n=1}^{\infty} \Delta\Phi(Z)_{T_n} \int_{(0,s]} \delta_{T_n}(du) \tag{A.1}$$

Therefore, for $s > 0$, $dZ_s = dX_s + d(\rho-\lambda)_s + \sum_{n=1}^{\infty} \Delta Z_{T_n}\delta_{T_n}(ds)$. As $Z_s^c = x + \sigma W_s$, we have $d\langle Z^c, Z^c\rangle_s = d\langle \sigma W, \sigma W\rangle_s = \sigma^2 d\langle W, W\rangle_s = \sigma^2 ds$. Therefore,

$$\Phi(Z_t) = \Phi(Z_0) + \int_{(0,t]} D\Phi(Z_{s-})\left[\mu ds + \sigma dW_s + d(\rho-\lambda)_s + \sum_{n=1}^{\infty}\Delta Z_{T_n}\delta_{T_n}(ds)\right] + \frac{\sigma^2}{2}\int_{(0,t]} ds\, D^2\Phi(Z_{s-}) + \sum_{n=1}^{N(t)}[\Delta\Phi(Z)_{T_n} - D\Phi(Z_{T_n-})\Delta Z_{T_n}]$$

Note that

$$\begin{aligned}\int_{(0,t]}\left(\sum_{n=1}^{\infty}\Delta Z_{T_n}\delta_{T_n}\right)(ds)D\Phi(Z_{s-}) &= \sum_{n=1}^{\infty}\Delta Z_{T_n}\int_{(0,t]}\delta_{T_n}(ds)D\Phi(Z_{s-}) \\ &= \sum_{n=1}^{\infty}\Delta Z_{T_n}D\Phi(Z_{T_n-})1_{(0,t]}(T_n) \\ &= \sum_{n=1}^{N(t)}\Delta Z_{T_n}D\Phi(Z_{T_n-})\end{aligned}$$

Using this formula, cancelling terms, and using the fact that the continuity of integrators allows us to replace Z_{s-} by Z_s, we have

$$\Phi(Z_t) = \Phi(Z_0) + \sigma \int_{(0,t]} D\Phi(Z_s)dW_s + \int_{(0,t]} ds \left[\mu D\Phi(Z_s) + \frac{\sigma^2}{2} D^2\Phi(Z_s)\right] + \int_{(0,t]} D\Phi(Z_s)d(\rho - \lambda)_s + \sum_{n=1}^{N(t)} \Delta\Phi(Z)_{T_n}$$

It follows from (A.1) that $d\sum_{n=1}^{N(s)} \Delta\Phi(Z)_{T_n} = \sum_{n=1}^{\infty} \Delta\Phi(Z)_{T_n}\delta_{T_n}(ds)$. Integrating by parts (Elliott, 1982, Corollary 12.22), we have

$$\begin{aligned} e^{-\alpha t}\Phi(Z_t) &= \int_{(0,t]} e^{-\alpha s} d\Phi(Z,p)_s - \alpha \int_{(0,t]} ds\, e^{-\alpha s}\Phi(Z_s) + \Phi(Z_0) \\ &= \Phi(Z_0) + \int_{(0,t]} e^{-\alpha s}\left[\sigma D\Phi(Z_s)dW_s + ds\left(\mu D\Phi(Z_s) + \frac{\sigma^2}{2}D^2\Phi(Z_s)\right)\right. \\ &\quad \left. + D\Phi(Z_s)d(\rho-\lambda)_s + \sum_{n=1}^{\infty} \delta_{T_n}(ds)\Delta\Phi(Z)_{T_n}\right] - \alpha \int_{(0,t]} ds\, e^{-\alpha s}\Phi(Z_s) \\ &= \Phi(Z_0) + \sigma \int_{(0,t]} e^{-\alpha s} D\Phi(Z_s)dW_s + \int_{(0,t]} ds\, e^{-\alpha s}\Gamma\Phi(Z_s) \\ &\quad + \int_{(0,t]} e^{-\alpha s} D\Phi(Z_s)d(\rho-\lambda)_s + \int_{(0,t]} \left(\sum_{n=1}^{\infty} \Delta\Phi(Z)_{T_n}\delta_{T_n}\right)(ds)e^{-\alpha s} \end{aligned}$$

The last formula can be re-written as

$$\sum_{n=1}^{\infty} \Delta\Phi(Z)_{T_n} \int_{(0,t]} \delta_{T_n}(ds)e^{-\alpha s} = \sum_{n=1}^{\infty} \Delta\Phi(Z)_{T_n} e^{-\alpha T_n} 1_{(0,t]}(T_n) = \sum_{n=1}^{N(t)} \Delta\Phi(Z)_{T_n} e^{-\alpha T_n}$$

Given our assumptions, it follows (Karatzas & Shreve, Proposition 2.10) that the stochastic integral

$$\left(\int_{(0,t]} e^{-\alpha s} D\Phi(Z_s)dW_s\right)_{t\in\Re_+}$$

is a martingale. As $T_0 = 0$, it follows that $\Phi(Z_0) = \Phi(x) + \Delta\Phi(Z)_{T_0}$. Therefore, taking expectations in the above equation yields the result. ■

Proof of Lemma 2.2.2. We have

$$\begin{aligned} C_t(R,L;x,\mu,\sigma,h,0,l,\alpha) &= h\int_{[0,t]} ds\, e^{-\alpha s} Z_s + l \int_{[0,t]} ds\, e^{-\alpha s} dL_s \\ &= h\int_{[0,t]} ds\, e^{-\alpha s}(x + \mu s + \sigma W_s + R_s - L_s) + l\int_{[0,t]} ds\, e^{-\alpha s} dL_s \end{aligned}$$

Elementary calculations yield

$$\int_{[0,t]} ds\, e^{-\alpha s} = \frac{1}{\alpha}(1 - e^{-\alpha t}) \qquad \text{and} \qquad \int_{[0,t]} ds\, e^{-\alpha s} s = \frac{1}{\alpha^2}(1 - e^{-\alpha t}) - \frac{1}{\alpha} t e^{-\alpha t}$$

Evaluating the fourth term, we have

$$\int_{[0,t]} ds\, e^{-\alpha s} R_s = \int_{[0,t]} ds\, e^{-\alpha s} \rho_s + \int_{[0,t]} ds\, e^{-\alpha s} \sum_{n=0}^{N(s)} \Delta R_{T_n}$$

As $\rho_0 = 0$, we have

$$\int_{[0,t]} ds\, e^{-\alpha s} \rho_s = \frac{1}{\alpha} \left(\int_{[0,t]} e^{-\alpha s} d\rho_s - e^{-\alpha t} \rho_t \right)$$

and

$$\begin{aligned}
\int_{[0,t]} ds\, e^{-\alpha s} \sum_{n=0}^{N(s)} \Delta R_{T_n} &= \int_{\Re_+} ds\, e^{-\alpha s} 1_{[0,t]}(s) \sum_{n=0}^{\infty} \Delta R_{T_n} 1_{[0,s]}(T_n) \\
&= \sum_{n=0}^{\infty} \Delta R_{T_n} \int_{\Re_+} ds\, e^{-\alpha s} 1_{[0,t]}(s) 1_{[T_n,\infty)}(s) \\
&= \sum_{n=0}^{\infty} \Delta R_{T_n} 1_{[0,t]}(T_n) \int_{\Re_+} ds\, e^{-\alpha s} 1_{[T_n,t]}(s) \\
&= \sum_{n=0}^{\infty} \Delta R_{T_n} 1_{[0,t]}(T_n) \int_{[T_n,t]} ds\, e^{-\alpha s} \\
&= \frac{1}{\alpha} \sum_{n=0}^{N(t)} \Delta R_{T_n} (e^{-\alpha T_n} - e^{-\alpha t})
\end{aligned}$$

as $N(t)$ is finite almost surely. Thus,

$$\begin{aligned}
\int_{[0,t]} ds\, e^{-\alpha s} R_s &= \frac{1}{\alpha} \left[\int_{[0,t]} e^{-\alpha s} d\rho_s - e^{-\alpha t} \rho_t + \sum_{n=0}^{N(t)} \Delta R_{T_n} (e^{-\alpha T_n} - e^{-\alpha t}) \right] \\
&= \frac{1}{\alpha} \int_{[0,t]} e^{-\alpha s} dR_s - \frac{1}{\alpha} e^{-\alpha t} R_t
\end{aligned}$$

The integral $\int_{[0,t]} ds\, e^{-\alpha s} L_s$ is manipulated analogously. Therefore,

$$\begin{aligned}
C_t(R, L; x, \mu, \sigma, h, 0, l, \alpha) = \left(\frac{hx}{\alpha} + \frac{h\mu}{\alpha^2} \right) (1 - e^{-\alpha t}) - \frac{h\mu}{\alpha} t e^{-\alpha t} + h\sigma \int_{[0,t]} ds\, e^{-\alpha s} W_s \\
+ \frac{h}{\alpha} \left[\int_{[0,t]} e^{-\alpha s} (dR - dL)_s - e^{-\alpha t} (R - L)_t \right] \\
+ l \int_{[0,t]} ds\, e^{-\alpha s} dL_s
\end{aligned}$$

Taking expectations and letting $t \uparrow \infty$, we have

$$
\begin{aligned}
&EC_\infty(R, L; x, \mu, \sigma, h, 0, l, \alpha) \\
&\qquad = \frac{hx}{\alpha} + \frac{h\mu}{\alpha^2} + \frac{h}{\alpha} E \int_{\Re_+} ds\, e^{-\alpha s} dR_s + \left(l - \frac{h}{\alpha}\right) E \int_{\Re_+} ds\, e^{-\alpha s} dL_s \\
&\qquad = \frac{hx}{\alpha} + \frac{h\mu}{\alpha^2} + EC_\infty(R, L; x, \mu, \sigma, 0, h/\alpha, l - h/\alpha, \alpha)
\end{aligned}
$$

which is the desired formula. ■

Proof of Lemma 2.3.1. Note that

$$
\begin{aligned}
&E\left[e^{-\alpha t}\Phi(Z_t) + C_t(R, L; x, \mu, \sigma, 0, \delta, l - \delta, \alpha) - \Phi(x) - \int_{(0,t]} ds\, e^{-\alpha s} \Gamma\Phi(Z_s)\right] \\
&\qquad = E\int_{(0,t]} e^{-\alpha s}[D\Phi(Z_s) + \delta] d\rho_s - E\int_{(0,t]} e^{-\alpha s}[D\Phi(Z_s) - (l - \delta)] d\lambda_s \\
&\qquad\quad + E\sum_{n=0}^{N(t)} e^{-\alpha T_n}[\Phi(Z_{T_n}) - \Phi(Z_{T_n} - \Delta R_{T_n}) + \delta \Delta R_{T_n}) \\
&\qquad\quad + E\sum_{n=0}^{N(t)} e^{-\alpha T_n}[\Phi(Z_{T_n}) - \Phi(Z_{T_n} + \Delta L_{T_n}) + (l - \delta)\Delta L_{T_n})] \\
&\qquad = E\int_{(0,t]} e^{-\alpha s}[D\Phi(Z_s) + \delta] d\rho_s - E\int_{(0,t]} e^{-\alpha s}[D\Phi(Z_s) - (l - \delta)] d\lambda_s \\
&\qquad\quad + E\sum_{n=0}^{N(t)} e^{-\alpha T_n} \int_{Z_{T_n} - \Delta R_{T_n}}^{Z_{T_n}} dy\, [D\Phi(y) + \delta] \\
&\qquad\quad - E\sum_{n=0}^{N(t)} e^{-\alpha T_n} \int_{Z_{T_n}}^{Z_{T_n} + \Delta L_{T_n}} dy\, [D\Phi(y) - (l - \delta)]
\end{aligned}
$$

Our hypotheses imply that

$$
E[e^{-\alpha t}\Phi(Z_t) + C_t(R, L; x, \mu, \sigma, 0, \delta, l - \delta, \alpha) - \Phi(x)] \geq 0 \tag{A.2}
$$

By the mean value theorem, $\Phi(Z_t) = \Phi(0) + D\Phi(c)Z_t$ for some $c \in [0, Z_t]$; consequently, $e^{-\alpha t}\Phi(Z_t) = e^{-\alpha t}\Phi(0) + e^{-\alpha t}D\Phi(c)Z_t$. Setting $\eta = \max\{\delta, |l - \delta|\}$, it follows from (a) that $|D\Phi(c)| \leq \eta$. As (R, L) is a feasible policy, it follows from Definition 2.1.3(a) that $Z_t \geq 0$ almost surely. Therefore, $|e^{-\alpha t}\Phi(Z_t)| \leq |e^{-\alpha t}\Phi(0)| + e^{-\alpha t}\eta Z_t$. Consequently, $Ee^{-\alpha t}|\Phi(Z_t)| \leq e^{-\alpha t}|\Phi(0)| + \eta E e^{-\alpha t} Z_t$. It follows from Definition 2.1.3(c) that $\lim_{t\uparrow\infty} Ee^{-\alpha t}|\Phi(Z_t)| = 0$. Thus, as $t \uparrow \infty$ the first term in (A.2) vanishes, thereby yielding the result. ■

Proof of Lemma 2.3.6. Suppose $\delta > l$. It is straightforward to check that the expression on the left-hand-side of (2.3.5) is a strictly decreasing function of $S \in \Re$. Moreover, this function is concave over $\Re_+$. Note that the value of this function at 0 is $2\gamma(l-\delta)$. If $l < \delta$, then $2\gamma(l-\delta) > -2\gamma\delta$. By the intermediate value theorem, there exists a unique S that solves (2.3.5).

Conversely, suppose $\delta \le l$. If $\delta = l$, then (2.3.5) has no solution. Suppose $\delta < l$. It is straightforward to check that the expression on the left-hand-side of (2.3.5) is a strictly increasing function of $S \in \Re$. Moreover, this function is convex over $\Re_+$. Note that the value of this function at 0 is $2\gamma(l-\delta) > 0 > -2\gamma\delta$. Thus, there is no $S > 0$ that solves (2.3.5). ■

Proof of Lemma 2.3.7. (A) follows from construction.

(B) Clearly, these inequalities hold on the set $\{0\} \cup [S, \infty)$.

Consider $(0, S)$. Let $g(.;S) = DF(.;S)$. Then, $\Gamma g(x;S) = 0$ for every $x \in (0, S)$, $g(0;S) = -\delta < l - \delta = g(S;S) < 0$. If $g(x;S) < -\delta$ for some $x \in (0, S)$, then there exists $x^* \in (0, S)$ such that $g(x^*;S) = \min_{x \in [0,S]} g(x;S) < -\delta$. Consequently, $Dg(x^*;S) = 0$ and $D^2 g(x^*;S) \ge 0$. It follows that $\Gamma g(x^*;S) > 0$, which is a contradiction. Thus, $DF(x;S) = g(x;S) \ge -\delta$ for every $x \in (0, S)$.

We now show that $DF(x;S) = g(x;S) \le l - \delta$ for every $x \in (0, S)$. Suppose there exists $x \in (0, S)$ such that $g(x;S) > l - \delta$. Then, there exists $x^* \in (0, S)$ such that $g(x^*;S) = \max_{x \in [0,S]} g(x;S)$. It follows that $Dg(x^*;S) = 0$. We also have $Dg(S;S) = D^2 F(S;S) = D^2 f(S;S) = 0$. Note that $\Gamma Dg(x;S) = 0$ for every $x \in (x^*, S)$. Suppose $\max_{x \in [x^*,S]} Dg(x;S) > 0$. Then, there exists $x^{**} \in (x^*, S)$ such that $\Gamma Dg(x^{**};S) < 0$, a contradiction. So, $Dg(x;S) \le 0$ for every $x \in [x^*, S]$. Similarly, $Dg(x;S) \ge 0$ for every $x \in [x^*, S]$. So, $Dg(x;S) = 0$ for every $x \in [x^*, S]$. It follows that $l - \delta = g(S;S) = g(x^*;S) + \int_{x^*}^{S} dy\, Dg(y;S) = g(x^*;S) > l - \delta$, which is a contradiction.

(C) For $x \in [0, S]$, $\Gamma F(x;S) = \Gamma f(x;S) = 0$. Consider $x > S$. Note that, as $D^2 f(S;S) = 0$, we have $\mu Df(S;S) - \alpha f(S;S) = \Gamma f(S;S) = 0$. Therefore,

$$
\begin{aligned}
\Gamma F(x;S) &= \mu(l-\delta) - \alpha[f(S;S) + (l-\delta)(x-S)] \\
&= \mu(l-\delta) - \mu Df(S;S) - \alpha(l-\delta)(x-S) \\
&= -\alpha(l-\delta)(x-S) \\
&> 0
\end{aligned}
$$

which conclues the proof. ■

Proof of Lemma 2.3.11. (A) Suppose (R, L) solves (2.3.10). It follows from (2.3.10) that $0 \leq Z_t = X_t + R_t - L_t \leq S$ for every $t \in \Re_+$. Consequently, conditions 2.1.3(a) and 2.1.3(c) are satisfied. Condition 2.1.3(b) will follow from the continuity of (R, L) and the conventions that $T_0 = 0$ and $\inf \emptyset = \infty$.

(i) It follows directly from (2.3.10) that R and L are non-negative and non-decreasing processes.

(ii) Consider $t \in \Re_+$. By (2.3.10), $R_t \geq L_t - X_t$ and $L_t \geq X_t + R_t - S$. If $R_t = L_t - X_t$, then $L_t > X_t + R_t - S$. Otherwise, $L_t = X_t + R_t - S$, which implies $S = 0$, a contradiction. Similarly, if $L_t = X_t + R_t - S$, then $R_t > L_t - X_t$.

(iii) If $R_0 = [L_0 - x]^+ > 0$, then $R_0 = L_0 - x$. By (ii), this means $L_0 > x + R_0 - S$. Since $L_0 = [x + R_0 - S]^+$, this means $L_0 = 0$. Therefore, $R_0 = -x \leq 0$, a contradiction. It follows that $R_0 = 0$ and $L_0 = [x - S]^+ \geq 0$.

(iv) Since, R and L are non-decreasing, their sample paths must have left-hand limits at every t, denoted by R_{t-} and L_{t-} respectively. Note that

$$R_{t-} = \lim_{n \uparrow \infty} R_{t-1/n} = \lim_{n \uparrow \infty} \sup \left\{ [L_u - X_u]^+ \mid u \in [0, t - 1/n] \right\}$$

Since

$$\sup \left\{ [L_u - X_u]^+ \mid u \in [0, t - 1/n] \right\} \leq \sup \left\{ [L_u - X_u]^+ \mid u \in [0, t) \right\}$$

for every $n \in \mathcal{N}$, we have

$$R_{t-} = \lim_{n \uparrow \infty} \sup \left\{ [L_u - X_u]^+ \mid u \in [0, t - 1/n] \right\} \leq \sup \left\{ [L_u - X_u]^+ \mid u \in [0, t) \right\}$$

Conversely, for every $n \in \mathcal{N}$, there exists $u_n \in (t - 1/n, t)$ such that

$$\sup \left\{ [L_u - X_u]^+ \mid u \in [0, t) \right\} - 1/n < [L_{u_n} - X_{u_n}]^+ \leq R_{u_n}$$

Letting $n \uparrow \infty$, we have

$$\sup \left\{ [L_u - X_u]^+ \mid u \in [0, t) \right\} \leq R_{t-}$$

As an analogous argument applies to L_{t-}, we have

$$R_{t-} = \sup \left\{ [L_u - X_u]^+ \mid u \in [0, t) \right\} \qquad \text{and} \qquad L_{t-} = \sup \left\{ [X_u + R_u - S]^+ \mid u \in [0, t) \right\}$$

(v) Suppose $R_t > R_{t-}$ and $L_t > L_{t-}$ for some $t \in \Re_+$. Then, using (iv),

$$R_t = \sup\left\{[L_u - X_u]^+ \mid u \in [0,t]\right\} > \sup\left\{[L_u - X_u]^+ \mid u \in [0,t)\right\} = R_{t-}$$

and

$$L_t = \sup\left\{[X_u + R_u - S]^+ \mid u \in [0,t]\right\} > \sup\left\{[X_u + R_u - S]^+ \mid u \in [0,t)\right\} = L_{t-}$$

Consequently, $R_t = [L_t - X_t]^+ > 0$ and $L_t = [X_t + R_t - S]^+ > 0$. It follows that $R_t = L_t - X_t$ and $L_t = X_t + R_t - S$, which contradicts (ii). So, $R_t > R_{t-}$ implies $L_t = L_{t-}$, and similarly, $L_t > L_{t-}$ implies $R_t = R_{t-}$, i.e., R and L cannot jump at the same t.

(vi) Let $t \in \Re_+$ be such that $R_t - R_{t-} > 0$; by (v), this implies $L_t = L_{t-}$. Then, $R_t = [L_t - X_t]^+ > 0$. Consequently, $R_t = L_t - X_t$ and for every $n \in \mathcal{N}$,

$$L_t - X_t = R_t > R_{t-} \geq [L_{t-1/n} - X_{t-1/n}]^+ \geq L_{t-1/n} - X_{t-1/n}$$

Therefore,

$$L_t - X_t > R_{t-} \geq \lim_{n\uparrow\infty}(L_{t-1/n} - X_{t-1/n}) = L_{t-} - X_{t-} = L_t - X_t$$

a contradiction. Thus, R is continuous on $\Re_+$. Similarly, L is continuous on $\Re_+$.

(B) We now define a control policy (R, L) that satisfies (2.3.10). Let $T_0 = 0$ and $(T_k)_{k\in\mathcal{N}}$ be an increasing positive sequence of stopping times. Let

$$T_1 = \inf\{t > 0 \mid X_t - [x-S]^+ \leq 0 \quad \vee \quad X_t - [x-S]^+ \geq S\}$$

Since $X_t - [x-S]^+$ is a continuous process, $X_{T_1} - [x-S]^+ \in \{0, S\}$. If $X_{T_1} - [x-S]^+ = 0$, then define (R, L) as follows:

$$R_t = \begin{cases} 0, & \text{if } t \in [T_0, T_1) \\ R_{T_{2k}}, & \text{if } t \in [T_{2k}, T_{2k+1}) \\ \sup\left\{[L_u - X_u]^+ \mid u \in [0,t]\right\}, & \text{if } t \in [T_{2k-1}, T_{2k}) \end{cases} \qquad (A.3a)$$

and

$$L_t = \begin{cases} [x-S]^+, & \text{if } t \in [T_0, T_1] \\ \sup\left\{[X_u + R_u - S]^+ \mid u \in [0,t]\right\}, & \text{if } t \in (T_{2k}, T_{2k+1}] \\ L_{T_{2k-1}}, & \text{if } t \in (T_{2k-1}, T_{2k}] \end{cases} \qquad (A.3b)$$

If $X_{T_1} - [x-S]^+ = S$, then define (R, L) as follows:

$$R_t = \begin{cases} 0, & \text{if } t \in [T_0, T_1] \\ R_{T_{2k-1}}, & \text{if } t \in (T_{2k-1}, T_{2k}] \\ \sup\{[L_u - X_u]^+ \mid u \in [0,t]\}, & \text{if } t \in (T_{2k}, T_{2k+1}] \end{cases} \qquad (A.4a)$$

and

$$L_t = \begin{cases} [x-S]^+, & \text{if } t \in [T_0, T_1] \\ \sup\{[X_u + R_u - S]^+ \mid u \in [0,t]\}, & \text{if } t \in (T_{2k-1}, T_{2k}] \\ L_{T_{2k-1}}, & \text{if } t \in (T_{2k}, T_{2k+1}] \end{cases} \qquad (A.4b)$$

Given (R, L), define $Z_t = X_t + R_t - L_t$. We have already specified T_0 and T_1. Specify the other stopping times as follows: given $k \in \mathcal{N}$, let

$$T_{2k} = \begin{cases} \inf\{t > T_{2k-1} \mid Z_t \geq S\}, & \text{if } Z_{T_1} = 0 \\ \inf\{t > T_{2k-1} \mid Z_t \leq 0\}, & \text{if } Z_{T_1} = S \end{cases} \qquad (A.5a)$$

and

$$T_{2k+1} = \begin{cases} \inf\{t > T_{2k} \mid Z_t \leq 0\}, & \text{if } Z_{T_1} = 0 \\ \inf\{t > T_{2k} \mid Z_t \geq S\}, & \text{if } Z_{T_1} = S \end{cases} \qquad (A.5b)$$

We now show that (R, L), defined by (A.3), (A.4) and (A.5), satisfies (2.3.10).

We first show that (2.3.10) holds for $t \in [0, T_1]$. Suppose there exists $t \in [0, T_1)$ such that $0 = R_t \neq \sup\{[L_u - X_u]^+ \mid u \in [0,t]\}$. It follows that, for some $u \in [0,t]$, $[L_u - X_u]^+ > 0$, i.e., $[x-S]^+ - X_u = L_u - X_u > 0$, but this contradicts the definition of T_1. Similarly, suppose there exists $t \in [0, T_1)$ such that $[x-S]^+ = L_t \neq \sup\{[X_u + R_u - S]^+ \mid u \in [0,t]\} = \sup\{[X_u - S]^+ \mid u \in [0,t]\}$. It follows that, for some $u \in [0,t]$, $[X_u - S]^+ > [x-S]^+ \geq 0$, i.e. $X_u - S > [x-S]^+$, but this contradicts the definition of T_1.

We now show that (2.3.10) holds for intervals of the form $(T_{2k}, T_{2k+1}]$. For $t \in (T_{2k}, T_{2k+1}]$, (2.3.10b) holds by definition. Suppose there exists $t \in (T_{2k}, T_{2k+1}]$ such that (2.3.10a) does not hold, i.e., $R_{T_{2k}} = R_t \neq \sup\{[L_u - X_u]^+ \mid u \in [0,t]\}$. It follows from (A.3a) that $0 \leq R_{T_{2k}} < [L_u - X_u]^+$ for some $u \in (T_{2k}, t]$. It follows that $L_u - X_u = [L_u - X_u]^+ > R_{T_{2k}}$, which implies $X_u + R_u - L_u = X_u + R_{T_{2k}} - L_u < 0$, a contradiction of the definition of T_{2k+1}. An analogous proof can be given for intervals of the form $(T_{2k-1}, T_{2k}]$.

Finally, we show that the solution constructed above is unique. Suppose (R, L) and (R', L') are distinct solutions of (2.3.10). Let $\omega \in \Omega$ be such that $T(\omega) = \inf\{t \in \Re_+ \mid R_t(\omega) > R'_t(\omega)\} < \infty$. By definition, $R_t(\omega) = R'_t(\omega)$ for every $t \in [0, T(\omega))$. Consequently,

$L_t(\omega) = L'_t(\omega)$ for every $t \in [0, T(\omega))$. By the continuity of $R(\omega)$ and $R'(\omega)$, this implies $R_{T(\omega)}(\omega) = R'_{T(\omega)}(\omega)$ and $L_{T(\omega)}(\omega) = L'_{T(\omega)}(\omega)$. Also, by continuity, there exists $\epsilon > 0$ such that $R_t(\omega) > R'_t(\omega)$ for every $t \in (T(\omega), T(\omega) + \epsilon)$. Consequently,

$$\begin{aligned} L_t(\omega) &= \sup\left\{[X_u(\omega) + R_u(\omega) - S]^+ \mid u \in [0,t]\right\} \\ &\geq \sup\left\{[X_u(\omega) + R'_u(\omega) - S]^+ \mid u \in [0,t]\right\} \\ &= L'_t(\omega) \end{aligned}$$

for every $t \in (T(\omega), T(\omega) + \epsilon)$. This means $R_{T(\omega)}(\omega) = L_{T(\omega)}(\omega) - X_{T(\omega)}(\omega)$. By (i), $L_{T(\omega)} > X_{T(\omega)}(\omega) + R'_{T(\omega)}(\omega) - S$. It follows that, for some $0 < \delta \leq \epsilon$, $L_t(\omega) = L_{T(\omega)}(\omega)$ for every $t \in (T(\omega), T(\omega) + \delta)$. Consequently, $L_t(\omega) \geq L'_t(\omega) \geq L'_{T(\omega)}(\omega) = L_{T(\omega)}(\omega) = L_t(\omega)$ for every $t \in (T(\omega), T(\omega) + \delta)$. Since $L_t(\omega) = L'_t(\omega)$ for every $t \in (T(\omega), T(\omega) + \delta)$, we have $R_t(\omega) = R'_t(\omega)$ for every $t \in (T(\omega), T(\omega) + \delta)$, a contradiction.

(C) By Lemma 2.2.1, the definition of C_t, and the definition of (R, L),

$$\begin{aligned} E[e^{-\alpha t}&F(Z_t) + C_t(R, L; x, \mu, \sigma, 0, \delta, l - \delta, \alpha)] \\ &= F(x) + E\int_{(0,t]} e^{-\alpha s}[DF(Z_s) + \delta]d\rho_s \\ &\quad - E\int_{(0,t]} e^{-\alpha s}[DF(Z_s) - (l - \delta)]d\lambda_s + E\int_{(0,t]} ds\, e^{-\alpha s}\Gamma F(Z_s) \\ &\quad + E\Delta F(Z)_{T_0} + E(l - \delta)\Delta L_{T_0} \end{aligned} \tag{A.6}$$

Consider the right-hand-side of (A.6). By the definition of (R, L), $d\rho_s > 0$ if and only if $Z_s = 0$, and $d\lambda_s > 0$ if and only if $Z_s = S$. Since $DF(0) = -\delta$ and $DF(S) = l - \delta$, the second and third terms vanish. As $Z_s \in [0, S]$ for every $s > 0$, we have $\Gamma F(Z_s) = 0$ for every $s > 0$. Therefore, the fourth term vanishes. The last two terms can be written as $E[F(Z_0) - F(x) + (l - \delta)\Delta L_{T_0}]$. If $x \in [0, S]$, then $Z_0 = x$ and $\Delta L_{T_0} = 0$; consequently, $E[F(Z_0) - F(x) + (l - \delta)\Delta L_{T_0}] = 0$. If $x > S$, then $Z_0 = S$ and $\Delta L_{T_0} = x - S$; consequently, $E[F(Z_0) - F(x) + (l - \delta)\Delta L_{T_0}] = E[F(S) - F(x) + (l - \delta)(x - S)] = 0$. Thus, we have

$$E[e^{-\alpha t}F(Z_t) + C_t(R, L; x, \mu, \sigma, 0, \delta, l - \delta, \alpha)] = F(x)$$

By construction, $Z_t \in [0, S]$ for every $t \in \Re_+$. As $[0, S]$ is compact and F continuous, $\{F(Z_t) \mid t \in \Re_+\}$ is bounded. Therefore, as $t \uparrow \infty$, the first term vanishes, yielding the desired result. ■

Proof of Lemma 3.2.2. Suppose $D_{12}c > 0$. The other case can be handled analogously.

Suppose $U(\alpha, \alpha) \geq U(\alpha, \alpha')$ for all α, $\alpha' \in A$. It follows that $D_2U(\alpha, \alpha) = 0$ for almost every $\alpha \in A$. Adding the inequalities for types α and α', and cancelling common terms, we have

$$\int_{e(\alpha)}^{e(\alpha')} de \int_{\alpha'}^{\alpha} dx\, D_{12}c(x, e) \geq 0$$

If $\alpha > \alpha'$, then $e(\alpha') \geq e(\alpha)$, i.e., e is non-increasing.

Conversely, suppose $D_2U(\alpha, \alpha) = 0$ for almost every $\alpha \in A$ and e is non-increasing, and there exist α, $\alpha' \in A$ such that $U(\alpha, \alpha) < U(\alpha, \alpha')$. This amounts to

$$0 < \int_{\alpha}^{\alpha'} dx\, D_2U(\alpha, x) = \int_{\alpha}^{\alpha'} dx\, [D_2U(\alpha, x) - D_2U(x, x)] = \int_{\alpha}^{\alpha'} dx \int_{x}^{\alpha} dy\, D_{12}U(y, x) \tag{A.7}$$

It is straightforward to check that $D_{12}U(y, x) = -D_{12}c(y, e(x))De(x) \geq 0$.

Suppose $\alpha' > \alpha$. Consider $x \in [\alpha, \alpha']$. It follows that $\int_{\alpha}^{\alpha'} dx \int_{x}^{\alpha} dy\, D_{12}U(y, x) \leq 0$, which contradicts (A.7).

Suppose $\alpha' < \alpha$. Consider $x \in [\alpha', \alpha]$. It follows that $\int_{\alpha}^{\alpha'} dx \int_{x}^{\alpha} dy\, D_{12}U(y, x) \leq 0$, which contradicts (A.7). ■

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