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When to Start a New Firm?

Modelling the Timing of Novice and Serial Entrepreneurs

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Abstract

The success of new start-up firms often depends on timing. It is valuable for the potential entrepreneur to wait for the right moment before starting a new firm. In this paper we provide a theoretical model to determine the optimal time for starting a new firm. We integrate insights from the real option theory with the theory on entrepreneurial market entry. An important and novel feature of our model is that it allows the start-up timing decisions of novice and serial entrepreneurs to be distinguished.

Keywords: entrepreneurship, serial entrepreneurship, start-ups, real options, stochastic optimal control

JEL classification: D92, D81, L26, M3

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1 Introduction

Entrepreneurial market entry through the start-up of new firms, has generated a substantial literature. This literature has variously been concerned about the determinants of market entry (e.g. Banerjee and Newman, 1993; Evans and Jovanovic, 1989; Fonseca et al., 2001), how the personal characteristics of latent, novice and habitual entrepreneurs influence market entry (e.g. Reynolds, 1997; Ucbasaran et al., 2006), as well as how market exit relates to market entry (e.g. Bosma et al., 2005). However, there has been little formal modelling of the optimal timing of starting a firm. Most of this research is empirical, and approaches entrepreneurship from the point of occupational choice theory. In practice, the success of new firms may depend on *when* they were started. Given that entrepreneurs will be aware of the importance of the timing of a start-up, there will clearly be instances where it is valuable for the entrepreneur to wait before starting up a new firm.

Although there is little formal modelling of this idea in the entrepreneurship literature, it has not been completely neglected in the market entry literature, the notable exception being Dixit (1989) who uses an option pricing approach to show that when a firm considers undertaking a project, waiting to do so has an option value. Later Dixit and Pindyck (1994) developed a formal approach of investment decision under uncertainty which today is widely used in decision making. Apart from a large number of contributions on option theory in the finance literature, real option theory is presently diffusing as a general decision theory into several fields of economics and business. Just to mention fields like the migration decision[1], R&D investments[2], FDI, or cross-border acquisition and joint ventures decisions[3], technology positioning investments[4], the impact of taxes[5], and human capital and education decisions[6]. The major advantage of this approach is the evaluation of uncertainty and irreversibility of a decision. In our approach we extend the original Dixit-Pindyck investment model by including a planning and investment period. We determine the optimal time to start a new firm, and the length of the resulting planning and gestation period, and we providde a detailed analytical discussion of the determinants of the optimal timing. The role of entrepreneurial ability for entrepreneurship and learning from experiences can be addressed by looking at serial entrepreneurs and the effects of learning from experiences on timing of a sequence of market entry.

[1] See Burda (1993).

[2] See e.g. Paxon (2003), Miltersen and Schwartz (2004) and Schwartz (2004).

[3] See e.g. Rugman and Li (2005) and Gilroy and Lucas (2006).

[4] See e.g. McGrath (1997).

[5] See e.g. Niemann (1999), Agliardi (2001), Panteghini (2001) Niemann and Sureth (2004), or Panteghini and Schjelderup (2005).

[6] See. e.g. Hogan and Walker (2007), Jacobs (2007).

The remainder of the paper will proceed as follows. In section 2 we review the major concepts used in this paper and we highlight the distinction between novice and serial entrepreneurs. Latent entrepreneurs most often wait before starting their first firm, or re-starting a next firm if they had previously owned a firm. Some may indeed wait so long that they never get to establish a new or another firm. What determines this duration and how can the optimal duration be established? In section 3 of this paper we provide an answer to this question, by modelling the decision when to start a firm as analogous to an investment decision under uncertainty. For this we propose a *real option dynamic programming* model which includes the entrepreneur. Entrepreneurial ability, which we will describe in section 4, is an important feature in the modelling of entrepreneurship. Serial entrepreneurs are distinguished as entrepreneurs with the benefit of having had the opportunity to learn from their experience and thus build on their entrepreneurial ability. The paper concludes (in section 5) with a short summary, a discussion of the policy implications and potential extensions of the model.

2 A Review of Concepts and Definitions

The Entrepreneur

The formal modelling of the entrepreneur in economics has been complicated due to the wide range of views on the definition and concept of entrepreneurship (Coyne and Leeson, 2004:235). In this paper we therefore proceed by narrowly defining **entrepreneurship** as the 'process of starting and continuing to expand new businesses ' (Hart, 2003:5). The role of the entrepreneur in economic development and structural change is discussed in Naudé (2008) and Gries and Naudé (2008).

The process of entrepreneurship goes through at least four, but often five to six phases. These are the *conception phase* (when the latent entrepreneur perceives an opportunity), the *gestation phase* (when the opportunity is evaluated), the *infancy phase* (when the firm is created), and the *adolescence phase* (where the firm matures) (Reynolds 1993). A fifth phase is when the entrepreneur *exit* the market, either voluntary (by selling the firm or passing it on to the next generation) or involuntary (when the firm goes bankrupt), and a sixth phase is when the entrepreneur starts over - by starting up a new firm. Not all entrepreneurs who exit from owning a firm choose to begin over and start-up a new firm, but a sizeable number do. They are known as '*serial entrepreneurs*', or 'renascent entrepreneurs' (Stam et al. 2007a). In this paper our concern is with the length of the conception and gestation phases, as applied to both novice and serial entrepreneurs.

A **latent entrepreneur** is a person who would prefer to be self-employed and who is considering seeking or is actively seeking the opportunity (Blanchflower et al. 2001:680). In the OECD, about 25 per cent of the labour force has been found to be latent entrepreneurs (ibid.). Once they are actively trying to start up a business, they are described as **nascent entrepreneurs** (Robson

2007:865).

A **novice entrepreneur** is someone whose current firm is his or her first start-up. A novice entrepreneur can be contrasted with a **serial entrepreneur,** who can be defined as individuals who have '*sold or closed at least one business in which they had a minority or majority ownership stake, and currently have a minority or majority ownership stake in a single independent business*' (Ucbasaran et al., 2006:5). More generally they are studied under the broader heading of **habitual entrepreneurs**, which also includes **portfolio** entrepreneurs (persons who own and operate more than one firm at the same time). In this paper however, we confine our attention to the serial entrepreneur and leave the modeling of portfolio entrepreneurship for a possible future paper.

Sometimes entrepreneurs are also defined or described according to their reason for having started a firm. Thus one can distinguish between necessity entrepreneurs, evasive entrepreneurs and opportunity entrepreneurs. **Necessity entrepreneurs** are self-employed because of a lack of suitable wage employment, **evasive entrepreneurs** are in business to overcome regulations or avoid taxes, while **opportunity entrepreneurs** are self-employed by choice, in order to exploit some perceived 'opportunity' (see e.g. Henrekson 2007; Coyne and Leeson 2004).

Entrepreneurial ability plays a key role in the market entry decision, and the subsequent success of a start-up, as we will argue in the next section. As recognized by Kanniainen and Poutvaara (2007:676) '*people differ substantially in terms of their ability to produce a business idea, elaborate their idea, and make its way to a marketable product or service*'. Key entrepreneurial abilities are the alertness to perceive and act on opportunities (Licht 2007; Gaglio and Katz 2001), the ability to function under uncertainty and risk (Kihlstrom and Laffont 1979) and the ability to coordinate and manage a firm (Lucas, 1978).

Given that entrepreneurs deal with uncertainty and irreversibility, **real option theory** offers a potentially useful angle from which to approach the market entry decision. Because future profits of the new firm are fundamentally uncertain and determined by the time path of a random process, in this paper we draw on real option theory to better understanding the optimal timing for market entry.

Market Entry as Occupational Choice

Since we are concerned to model the start-up (market entry) decision of an entrepreneur, and to make a distinction between novice and serial entrepreneurs, it is useful to give a brief overview of the existing literature to show where we extend and complement this literature.

Economic theory has approached the market-entry decision of an individual as an occupational choice between self-employment or entrepreneurial activities, and wage-employment. Important contributions in this regard were made by amongst others Lucas (1978), Evans and Jovanovic (1989), and Murphy et al. (1991). Herein and in subsequent research the factors that determine this occupational choice depend broadly on an individual's entrepreneurial ability, the relative rates of return to entrepreneurship, obstacles such as capital constraints and entry (start-up) costs, and factors that influence the opportunity costs of becoming self-employed or an entrepreneur. A substantial empirical literature explores the determinants of market entry and the obstacles faced by entrepreneurs. In summary, one can distinguish between studies that attempt to identify macro-level constraints or determinants (e.g. Highfield and Smiley, 1987), those focusing on industry-level and firm-level constraints (e.g. Reynolds, 1992), on start-up costs and regulatory barriers (e.g. Fonseca et al., 2007) and those that attempt to test the predictions from theoretical models on the relationship between wealth, inequality, credit market constraints and market entry (e.g. Lloyd-Ellis and Bernhardt, 2000).

In simple terms, the occupational choice can be explained by explicitly incorporating entrepreneurial ability (which is often denoted by θ) into the production function, as in Murphy et al. (1991):

$$Q = A\theta F(L) - w.L \tag{i}$$

Where Q is output, A is a commonly available technology, $F(L)$ the relation between output and labour inputs, and w is the wage rate. Because each person has an entrepreneurial ability (some better than others), it can be shown, generalizing from Murphy et al. (1991) that a person will become an entrepreneur if profits and the non-pecuniary benefits from self-employment or entrepreneurial activities exceed wage income plus the additional benefits from being in wage employment.

$$(A\theta F(L(A)) - wL(A)) + \eta > wL(A) + C \tag{ii}$$

where η denotes the non-pecuniary benefits of entrepreneurship (following Blanchflower and Oswald 1998) and C start-up (sunk) costs, taxes and other diverse benefits from wage employment.

Equation (ii) summarizes the key economic determinants of entrepreneurial start-ups. On the left hand side we first find entrepreneurial ability (θ). Entrepreneurial ability is a core element of occupational choice models (e.g. Lucas 1978; Evans and Jovanovic 1989). Because entrepreneurial ability determines the marginal production from capital and labour (as in equation (i)) the size of the firm (proxied by the size L) will be determined by the extent to which the ability of the entrepreneur can be 'stretched' across greater number of employees (Fonseca et al. 2007).

On the right hand side of (ii) we have wages. Higher relative expected wages can be expected to lower the probability of an individual opting for self-employment or entrepreneurial activities. However, empirical research has noted a 'paradox' in that individuals often appear make the occupational choice in favour of self-employment or entrepreneurial activities when the monetary returns are less than they would have obtained if they had remained in or chosen wage employment (Hamilton 2000). Moskowitz and Vissing-Jorgensen (2002) offer a number of explanations, namely that these individuals have a high tolerance for risk, that they may misperceive risk, and are overly optimistic (see also Arabsheibani et al. 2000), or that there are large non-pecuniary benefits (θ in equation (ii)) to being an entrepreneur.

Education and experience have a theoretically ambiguous effect on start-ups, as it can influence both θ and w (Giannetti and Simonov 2004). It can raise the rate of start-ups as it improves entrepreneurial ability (Stam et al. 2007b:7). However it may also reduce the probability of self-employment or entrepreneurial activities, and raise the probability of firm exit as it raises the wage rate that an individual can earn in formal employment. In the context of serial entrepreneurship it is also been found that more educated persons are less likely to immediately re-start a new firm, as they have more opportunities in the labour market.

Finally, in (ii) we have start-up costs. It includes a fixed cost/sunk cost element such as planning and preparation, and the regulations that need to be adhered to in terms of labour and production and organisation standards (Fonseca et al. 2001). Entry costs and regulations—especially labour market regulations—tend to lower the start-up rate of new firms (e.g. Fonseca et al. 2001; Klapper et al. 2006).

In order to overcome start-up costs and investment sunk costs when starting up a new firm, entrepreneurs generally require access to capital. Following Stiglitz and Weiss (1981) it has been realized that capital markets could provide inadequate finance to entrepreneurs due to moral hazard and limited liability problems (Paulson et al. 2006). The observation that entrepreneurs are wealthier than wage-earners (e.g. Cagetti and De Nardi, 2005) has been taken as evidence of such capital constraints. Wealthier individuals are not only more likely to start-up new firms as wealth allows them to overcome start-up costs, but also if they need to earn more income in order to achieve similar utility from their income than less wealthy persons (see Newman, 2007).

The occupational choice model has been applied to both the market entry decisions of novice and serial entrepreneurs. In the case of serial entrepreneurs however, there has been a growing number of empirical studies which investigate how the determinants of serial entrepreneurship differ from those of novice entrepreneurs, what the characteristics of serial entrepreneurs are, and whether or not serial entrepreneurs perform better than novice entrepreneurs. A good overview of the current state of the literature on serial entrepreneurs (and more generally habitual entrepreneurs) is provided by Ucbasaran et al.(2006).

A review of this literature falls outside the scope of the present paper. For present purposes, it is sufficient to note that the fundamental differences between serial entrepreneurs and novice entrepreneurs are that the former have experience, a reputation, and may be older. This might influence the timing of their next market entry, depending on the nature of their experience and reputation. For instance, latent entrepreneurs who have previously successfully owned and managed a firm, and voluntarily exited the market, may wait less longer than novice entrepreneurs before re-entering the market, as they may be more confident about their abilities, may be better in handling risk/uncertainty, may have better established networks, be better able to spot opportunities, be able to plan more quickly, and because of reputational effects find it easier to obtain funding. In contrast latent entrepreneurs who have previously owned and managed a firm that was ultimately unsuccessful and was forced to exit the market, may have a longer waiting period as they may want to invest more in improving their entrepreneurial ability, engage in longer preparation, and may due to reputational effects and social norms and culture face greater difficulty to obtain funding. In both of these cases age, associated with a shorter planning horizon, may bring forward the decision to re-enter the market.

The occupational choice approach to market entry as summarised above thus in essence predicts that entrepreneurial market entry will occur if the benefits of market entry exceed the costs. The approach has nothing to offer explicitly on the timing of market entry - it is basically assumed to occur if (ii) is met. As it assumed that individuals continuously make the assessment of costs and benefits, different times of entry are assumed to reflect differences in individual circumstances. It does not take into account that when if (ii) is met, it may not be the optimal time for an individual to enter the market due to uncertainty of returns. Generally, the occupational choice approach treats uncertainty in an unsatisfactory manner. It is most often assumed that $A.F(L)$ is certain. If however, $A.F(L)$ is subject to uncertainty, and there is positive sunk costs in starting up a firm, the option to wait before market entry will be valuable. This is the essence of the approach to market entry and exit pioneered by Dixit (1989).

Market Entry as Stopping Rule and the Real Option Decision

Dixit (1989) does not explicitly consider entrepreneurs and entrepreneurial entry, but rather the decision of an existing firm to invest or not to invest in a new project, or to abandon an exiting project. He shows that if the returns for a risk-neutral firm of investing in a new project are subject to uncertainty and there is even a small amount of sunk costs, it will be costly for firms to reverse their decision. In such cases the option to wait and see if better returns can be obtained at a later date, both for firms contemplating market entry and exit[7], takes on a positive value. Dixit (1989) argues that this option value is significant even if sunk costs are relatively small. As such the start-up of a new firm can be modelled by following the Dixit and Pindyck (1994) approach.

Apart from a large number of contributions on option theory in the finance literature, real option theory is presently diffusing as a general decision theory into many fields of economics and business, as we remarked in the introduction. The major advantage of this approach is the evaluation of uncertainty and irreversibility of a decision. In this decision problem, time can move in only one direction (something is irreversibly done or not done with all related consequences). The decision is made by considering a stochastic dynamic environment and hence the relevant variables are described by random processes. Therefore, the approach goes beyond static stochastic decision problems and is

[7] A firm with an existing project that is performing unsatisfactorily will often continue with that project ('hysteresis') when sunk costs have been incurred, because it would be costly to re-enter the market if "future developments turn favourable" (Dixit, 1989:629).

able to model timing decisions[8]. As the decision problem is explicitly imbedded in a randomly changing dynamic environment, random changes in the environment would influence original timing decisions. For instance, even if an entrepreneur had expected to start a new firm at a particular future date, random changes in the environment could trigger - or delay - the start of the firm. In general, the approach is appropriate when there is uncertainty, irreversibility and a problem of optimal timing. As we argue in the next section, uncertainty and irreversibility are key characteristics of the entrepreneurial market entry decision. Adding these two elements to the entrepreneurial start-up problem helps to explain the optimal timing decision when to enter the market.

3 Optimal Timing to Start a Firm

The literature review in the previous section has distilled some of the stylized facts of what are known about the determinants of market entry of novice and serial entrepreneurs. It had shown that occupational choice modelling often abstracts from uncertainty and timing of market entry. In this section we propose a model to address these shortcomings. The market-entry decision at its heart is not just about the willingness to start a firm, but about the optimal time to do so. How long should an idea develop? How much time should be invested in planning and organizing market entry? When is the best time to enter the market?

Modelling the optimal timing of these decisions we commence this section by describing the uncertain future faced by the entrepreneur as consisting of various patterns of profit streams characterized by different stochastic processes - different for differing business ideas (products and services) and different entry periods. We then derive a price or profit threshold at which it is optimal for a latent entrepreneur to conclude the planning and waiting period (gestation) and enter the market (begin the infancy phase). Knowing this threshold we can then determine the expected duration of the planning and waiting period.

This optimal timing decision has three elements, namely *(i)* the accumulated *cost* and *benefits of planning and waiting,* the *(ii) value of the firm (expected net value of uncertain profits)* linked to the quality and nature of the product or service which the entrepreneur will offer on the market, and *(iii)* the *value of waiting* which includes the possibility of improving the quality and nature of the product or service, and the value of not being in an irreversible project, which is the *option value of the idea.*

3.1 Modelling the start-up decision under uncertainty

(i) Pre-Start-up Planning and Waiting (Conception and Gestation)

Before we turn to an explicit consideration of the specific conditions of a serial entrepreneur we start with a more general model of a start-up process. There

[8] It has also been used to explore timing decisions in education, see Bilkic/Gries/Pilichowski (2008).

are often substantial **costs** to starting up a new firm (Fonseca et al. 2001; Klapper et al. 2006). These include the costs required during the conception and gestation phases of the entrepreneurship process, such as market research, identification and evaluation of opportunities, establishment of networks and gaining of applicable knowledge. Let C_i represent these costs for a start-up firm i in each period. These costs may differ amongst entrepreneurs depending on their levels of entrepreneurial ability. Thus, at the end of the conception and gestation period, the same planning activities will have incurred different costs for different entrepreneurs. For simplicity these costs are constant for each start-up firm[9].

Therefore, if we define the time at which a latent entrepreneur perceives an opportunity as $t = 0$, the total investment is the sum of the costs of each period that the entrepreneur plans and waits until market entry occurs. We can see the total start-up costs as a sunk investment $I(T)$ which is dynamic: it increases over time with each additional period of planning and waiting. Denoting the end of the conception and gestation phase by T, the current value of total sunk investment in the start-up firm at the moment of market entry is:

$$I(T) = \int_0^T Ce^{r(T-t)}dt + \bar{C}, \tag{1}$$

where r is the risk-free interest rate. and $\bar{C}$ are the final major investments to realize the market entry. To focus on the major mechanics, taxes or subsidies are not included. However, this could be easily done by correcting the effective interest rates r, the costs of preparing[10] and the income streams for taxes.

However, waiting and planning also generates **benefits**. The latent entrepreneur can observe the market performance of other products and the payoffs of product characteristics and qualities. This will allow him or her to judge the likely market reward for improving his product or service. Hence, the resources invested in this phase generate substantial product improvements - or perceptions of improvements through marketing and brand-building - which could lead to an increase in the expected achievable market price and profits. This mechanism thus creates a potential link between product innovation and product branding and entrepreneurial entry, which we leave for future work to model in greater depth. For the present we can describe the virtual stochastic path of the pre-start-up market value (price) while investing in an improved product or service during the conception and gestation phase. In continuous time this random process can be described as a Brownian motion

$$d\tilde{P}_i = \delta\tilde{P}_i + \sigma_i\tilde{P}_i dW \quad \text{for} \quad t < T. \tag{2}$$

In (2) δ is the expected rate of market reward for a marginal improvement of the product variety generated during a period of waiting and investing in

[9] However, with an increasing number of sequential firms per entrepreneur they may decrease for a serial entrepreneur as the entrepreneur benefits from learning by doing.

[10] If these costs are tax deductible.

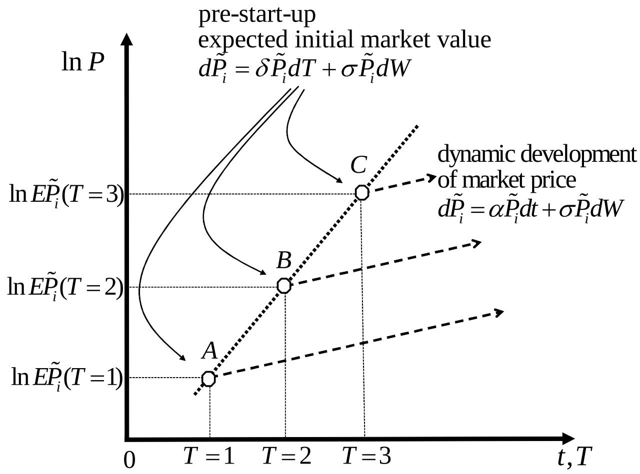

Figure 1: Pre-start-up expected initial market value, and market price dynamics

product improvements. It is the marginal pre-start-up market value differential with respect to product quality improvements (see in figure 1 the expected time path of the pre-start-up market value given by doted line)[11]. The expected marginal market reward δ must be sufficiently large to have a sufficient compensation for the additional investment costs of quality improvement C . σ_i denotes a constant volatility, and dW denotes the increments of a standard Wiener process.

(ii) Value of the Firm

Upon entry into market ($t > T$) the entrepreneur faces a stochastic revenue stream which is characterized by an expected average growth rate and elements of uncertainty. To keep matters simple, we assume that each start-up i is a price-taker offering a single well-defined product or service. The price of the product or service, $\{P_i(t)\}$ can be described by the geometric Brownian motion

$$dP_i = \alpha_i P_i dt + \sigma_i P_i dW \quad \text{for} \quad t > T, \tag{3}$$

with a constant drift α_i and a constant volatility σ_i, where $\alpha_i, \sigma_i > 0$. dW denotes the increments of a standard Wiener process; and α_i describes the deterministic part of the process. For $\alpha_i > 0$ the price P_i is expected to grow at the rate α_i. Depending on the start-up product i the individual price profiles show different growth patterns. As we assume that operating costs are constant, the price profile will be identical to the revenue and profit profile. This simplifies matters as we do not need to distinguish further between market prices, revenues and profits.

[11] For a formal discussion of the expected market value at market entry time T, see proposition 2.

The latent entrepreneur is assumed to be aware of the price and revenue profile of each potential start-up product i. The conception and gestation phase will come to an end and the entrepreneur will enter the market when a decision is taken to realize the specific expected revenue stream $\{P_i(t)\}$ associated with a particular start-up product i. In order to derive a rule for the optimal time to enter the market, we have to determine the expected value of the risky revenue stream for each potential start-up product i. Once a firm is started up, its product and service characteristics and hence the revenue profile are fixed. No other opportunities for business can be chosen and the entrepreneur is locked in. The economic value of the firm consists solely of its future revenue. For a risk neutral entrepreneur the gross *value of the start up* project V_i^{gross} is given by the expected present value of the *revenue stream* $\{P_i(t)\}$

$$V_i^{gross} = E(\int_T^{\infty} P_i e^{-r(t-T)} dt) = \frac{P_i}{r - \alpha_i}; \quad r > \alpha_i,$$

with r being the risk-free interest rate as opportunity costs.

For simplicity the entrepreneur has an infinite lifetime[12] and does not have the option of closing or selling his or her firm.[13] Having defined the gross value of the start-up above, we have to keep in mind that due to planning and preparation costs during the conception and gestation phase (1), the expected *gross value of the start-up* has to be adjusted for sunk investment $I(t)$. Hence, the net value of the revenue stream of the start up at the moment of market entry is

$$V_i = V_i^{gross} - I(T) \tag{4}$$

(iii) Option Value of Waiting

In addition to the expected *net value of the new firm* the third element of the decision problem, the *option value* of waiting and not committing to irreversible decisions has to be considered. This option not to irreversibly take the risk of failing has its own value - analogous to the investment decision facing a firm (Dixit, 1989) or the education decision facing an individual (Hogan and Walker 2007, Jacobs 2007). As long as the latent entrepreneur delays market entry he retains the option of market entry without the risk of failure and having to incur sunk investment costs. Waiting may open up additional opportunities

[12] Obviously the assumption of an infinite lifetime is not realistic. However, often entrepreneurs make family decisions. In this case entrepreneurs plan not only for themselves but also for the next or even more generations ahead. If we had assumed a finite lifetime, the end of the investment project would be determined by the end of the physical life of the entrepreneur. This date however, is random. While a random jump processes (modeled by Poisson processes) may address, this it will involve a substantial extension of our model. Since we wish to keep it simple for now, we leave the relaxation of this assumption for future work.

[13] Again, it is potentially possible to include exit options such as closing or selling the firm. In such a case a potential exit strategy becomes part of the original investment decision. However, given the additional complexity that this will introduce at this stage, we leave it for future work.

which could not have been foreseen and realized otherwise. Since market entry is regarded as irreversible, another period of waiting may enable the entrepreneur to obtain more information and reduce the risk of a failure or improve the market performance of the product. These additional opportunities imply that it is often beneficial to postpone market entry. Waiting is a value because it offers the latent entrepreneur flexibility.[14] Accounting for the option value F for the Brownian motion (2), the Hamilton-Jacobi-Bellman equation holds:

$$rFdt = E(dF). \tag{5}$$

This equation indicates that for a time interval dt, the total expected return on the investment opportunity is equal to the expected rate of capital appreciation.

3.2 Solving the Expected Time of Market Entry

Solving the optimal time of market entry as described above has two steps. First, for each potential start-up i we need to determine the price ($P_i^*(T)$ *threshold*) needed to enter the market. Second, the latent or potential entrepreneur will continually evaluate the price level. As soon as the threshold is reached, he or she will enter the market and start up a new firm. As long as the threshold is not reached the entrepreneur will keep on waiting. As long as the entrepreneur is waiting he or she will however make a prediction about the *expected timing to enter the market*. We will model these aspects in this section.

Decision Problem For an entrepreneur facing a number of periods over his or her lifetime, the decision whether or not to enter the market consists of evaluating the three elements introduced above, namely the start-up investment costs, the uncertainty of revenues and the option value of not entering the market. Given the expected *net value of the new business* (4), the *option value* F_i of waiting and postponing market entry (and improving the product) can be determined by applying dynamic programming.[15] Once the option value of waiting has been determined, the question whether or not to wait for another period will be determined by the solution to:

$$\max\{V_i(T), F_i(T)\} \tag{6}$$

At any time during the conception and gestation phase the latent entrepreneur will compare the expected *net value of the new firm* with the *option value* of remaining outside the market. As long as the option value of postponing market entry is higher than the value of realizing the uncertain revenue stream, the latent entrepreneur will opt for another period of waiting/planning. Solving this continuous decision problem determines the time of entry into the market

[14] Once the entrepreneur has decided to incur the sunk costs and enter the market, he or she could also decide to exit if the revenue that is realized is below expectation. However, as mentioned we assume for simplicitly that the entrepreneur cannot exit the market voluntarily after entering.

[15] See the next section.

and hence the optimal duration of the conception and gestation phase of the entrepreneurial process.

Determining the Entry Threshold

In order to determine the price that triggers market entry we need to consider the standard conditions concerning a stochastic dynamic programming problem of the introduced structure. In addition to the *Hamilton-Jacobi-Bellman equation* for the option value F_i and applying Ito's lemma to dF_i, we have to use the well known boundary conditions, namely (7), the *value matching condition (8),* and the *smooth pasting* condition (9)

$$F_\tau(0) = 0 \tag{7}$$

$$F_i(P^*) = V_i^{gross}(P^*) - I(T) \quad \text{value matching condition} \tag{8}$$

$$\frac{dF_i(P^*)}{dP} = \frac{dV_i(P^*)}{dP} \quad \text{smooth pasting condition} \tag{9}$$

to solve for the threshold market proce P^*. The setting of the decision problem implies that the value of the uncertain revenue stream must be worth the switch from waiting to market entry. Hence, the revenue level implied by the Brownian motion must be sufficiently high. This leads us to Proposition 1 as a point of departure:

Proposition 1 *For start- up costs of C per period and start-up investment costs $\bar{C}$, a pre-start-up market value following the Brownian motion 2, and a revenue stream that follows the Brownian motion 3 we can determine the threshold $P_i^*(T)$ that would trigger market entry as*

$$P_i^*(T) = \frac{\lambda_i}{\lambda_i - 1}(r - \alpha_i)\left[\frac{C}{r}\left(e^{rT} - 1\right) + \bar{C}\right], \tag{10}$$

$$\text{with} \quad \lambda = \frac{1}{2} - \frac{\delta}{\sigma^2} + \sqrt{(\frac{1}{2} - \frac{\delta}{\sigma^2})^2 + \frac{2r}{\sigma^2}}, \tag{11}$$

$$\text{and} \quad r > \delta \tag{12}$$

Proof: See Appendix 1.

Since each additional period of waiting increases the total start-up costs, total sunk investment costs increase over time and hence the entry-threshold becomes a function of T. To illustrate the threshold and its determinants figure 2a considers the case of start-up i. In figure 2a the horizontal axis represents time (t) and the vertical axis the natural log of the price (and revenue) level, P_i. Using logs the exponentially growing threshold (dashed line) can be drawn as a straight line instead of an exponential curve. In figure 2a the *threshold curve* is increasing over time because waiting is linked to additional product development and hence costs for improvements of the product. Figure 2a shows that an additional period of waiting drives up the threshold as the entrepreneur wants to be compensated for the additional costs by asking for a higher market price when entering the market.

Determining the Pre-start-up Expected Market Value

Once the entrepreneur knows from the threshold at which price level he or she should enter the market, the question is, when will this price threshold be realized? Waiting and planning results in substantial product improvements and increasing market rewards described by (2). Hence, we can now conclude that the *pre-start-up market value* and *revenue* from improving the product or service is an increasing function of the length of the conception and gestation period. This function can be represented by the *pre-start-up market value curve* $\tilde{P}_i(T)$ (dotted line) in figure 2a[16].

Proposition 2 *With the pre-start-up Brownian motion 2 we can determine the time path of the virtual expected pre-start-up value of the product* $\tilde{P}_i(T) = \tilde{P}_i(0)e^{((\delta-\frac{1}{2}\sigma^2)T+\sigma W(T))}$*, and hence determine the expected initial price for each duration of the conception and gestation period* T*.*

$$E\tilde{P}_i(T) = \tilde{P}_i(0)e^{\delta T}. \tag{13}$$

Proof. See Appendix 2.

Expected Time of Market Entry

The expected time of market entry can now be determined by comparing the current pre-start-up potential market value with the threshold. With the two curves, the *threshold* and the *pre-start-up expected market value curve,* we can determine the expected duration of the conception and gestation phase for each potential start-up i. The conception and gestation phase will end and the next phase of starting up will commence when the pre-start-up expected market value (revenue) reaches the threshold, given the present expectations for the development of the market value (revenue) during the waiting period, and given expectations about the firm's revenue profile.

Proposition 3 *a) With the threshold* $P_i^*(T)$ *and the pre-start-up expected start-up market value* $E\tilde{P}_i(T)$ *(see (2)), and condition (15) and (16) there exists an optimal time of market entry* $T^* = E(T) > 0$*. b) For each vector* $(\alpha, r, \sigma, T^*, C, \tilde{P}_i(0), \delta, \bar{C})$ *that fulfils a) there is a marginal environment, such that* T^* *is an implicit function of* $\alpha, \sigma, C, \tilde{P}_i(0), \delta, \bar{C}$ *and* r*.*

$$T^* = T^*(\alpha, \sigma, C, \tilde{P}_i(0), \delta\ r) \tag{14}$$

$$\frac{\lambda}{\lambda-1}(r-\alpha)\bar{C} \quad > \quad \tilde{P}_i(0), \tag{15}$$

$$\bar{C}r \quad > \quad \bar{C}\delta > C0) \tag{16}$$

[16] More precise, the log of the earning $\tilde{P}_i(T)$. See also appendix 2.

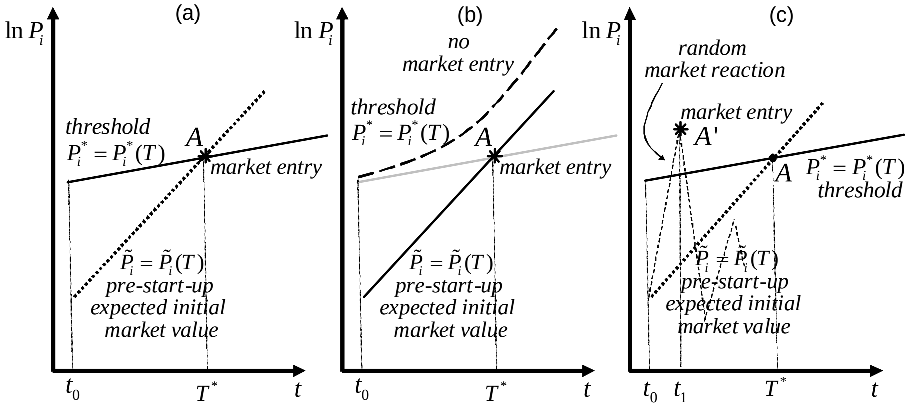

Figure 2: Threshold and Market Price Development

Proof. See Appendix 3.

In figure 2a, a higher level of the *threshold* compared to the *pre-start-up expected market values* indicates that the entrepreneurial efforts to develop the product during the conception and gestation phase (before T^*) are not yet sufficiently compensated by the present market value of the product. Hence he or she will not yet enter the market. In addition, the project may enter the market in finite time only if the slope of the pre-start-up expected market value curve is higher than the slope of the threshold. Extra time devoted to product improvements must have a positive net effect compared to the negative effects of postponing market entry (reflected by an increasing threshold). The optimal time of market entry is reached when the curves intersect and the required entry threshold is matched by the market value (price) of the product expected to be realized in the market.

If the market does not reward the costs of additional efforts sufficiently, market entry will not happen. This outcome is also illustrated by the dashed line in figure 2b. The increasing slope of the threshold curve indicates that additional time and costs spent on product development and placement must be compensated by sufficiently increasing market rewards and hence a rising expected market price at market entry. In figure 2b market rewards will not match the increase in the threshold. As a result there will be no intersect of the two curves and hence the entrepreneurial idea will not be realized as a start-up.

However, the expected time of market entry (T^*) is an indicator of what might happen in the future. When the conditions of the future are partly random, unexpected market entry can potentially occur at any time. In figure 2c we draw the time path of the pre-start-up expected market value of the product for the present state of information at time t_0 by the dotted line. Consequently, even if the entrepreneur expects to enter the market at T^*, a randomly occurring incident in the market can push the pre-start-up expected market price so that the threshold is reached and the business should be started immediately. In figure 2c this is displayed by the randomly upward shift of the realized market value at point A'. The observed and hence realized market value exceeds the threshold at t_1 and hence the business is started at t_1 and not -as expected before- at T^*. It is easy to find illustrative examples for such an unexpected early market entry If the entrepreneur is targeting a certain market niche, let us say a shop for high-heel shoes, and suddenly high-heels become fashionable, the entrepreneur will take the opportunity and immediately open his or her business. The entrepreneur will take this randomly occurring opportunity at t_1 - no matter what he planned and expected before.

This example shows that start-up decisions are timing decisions. Starting a business means *taking the opportunity at the right time*, even if the opportunity occurs accidentally.As the threshold and the pre-start-up expected market value curve are determined by a number of parameters, in the following section we will discuss how the optimal time of entry reacts when these different parameters change.

3.3 Determinants of the Expected Time of Market Entry

Having outlined the elements of the optimal start-up time in section 3.1, and having shown how the optimal time of market entry can be derived in section 3.2, the purpose of this section is to derive some comparative-statics and show how the model can be used to analyse the effect of various determinants on the timing of market entry. Without aiming to be exhaustive, we will analyse the impact of (i) price uncertainty σ_i and (ii) profit growth α_i, (iii) the path level of pre-start-up expected market values (revenues) $\tilde{P}_i(0)$, (iv) start-up costs C_i, (v) market rewards δ for improvements in product quality and variety, and (vi) interest rates r as opportunitiy costs. Figure 3 illustrates the reaction of market entry when different parameters are changing.

(i) Revenue Uncertainty

Proposition 4 *With an increase in risk of the future earnings σ_i expected market entry will be postponed, T^* will increase,*

$$\frac{dT^*}{d\sigma_i} = \underbrace{\frac{\left(e^{rT} - 1 + \frac{\bar{C}r}{C}\right)}{\left(e^{rT} - \frac{\delta\,(\lambda-1)}{(r-\alpha)\lambda}\frac{\tilde{P}_i(0)}{C}e^{\delta T}\right)}}_{<0} \frac{\overset{(-)}{\frac{\partial\lambda}{\partial\sigma}}}{\lambda\,(\lambda-1)\,r} > 0 \tag{17}$$

Proof. See Appendix 4.

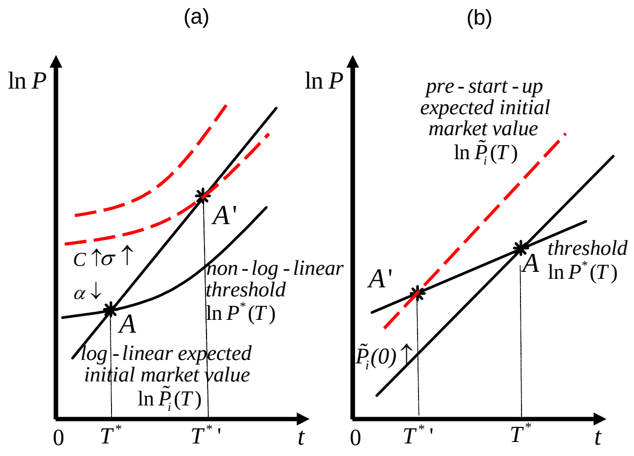

Figure 3: Changes in expected pre-start-up market evaluation

A negative effect of revenue risk - measured by the volatility of revenues - on the time of market entry is expected. An increasing risk of income will devalue the earning stream and hence decrease the attractiveness of opening the business. As long as additional net rewards of a longer phase of product development can compensate for the increase in the threshold, the project will still be pursued (see figure 3a). The increasing risk will shift the threshold curve upwards and hence will make a later market entry more attractive. Interestingly enough, this result is obtained even with risk neutral agents. We do not need to make any assumption about the utility function and risk aversion. The pure option value and the irreversibility include the effects of σ_i in a similar way.

If the project becomes too risky there is no intersect in figure 3a and the market entry is expected not to take place at all.

(ii) Revenue Growth

Proposition 5 *With an increase in the growth rate of the future earnings expected market entry T^* will speed up*

$$\frac{dT^*}{d\alpha} = \frac{[e^{rT} - 1 + \frac{\bar{C}_T}{C}]}{\underbrace{\left(e^{rT} - \frac{\delta\,(\lambda-1)}{(r-\alpha)\lambda}\frac{\tilde{P}_i(0)}{C}e^{\delta T}\right)}_{<0}} \frac{1}{(r-\alpha)r} < 0 \qquad (18)$$

Proof. See Appendix 5

A change in the revenue growth rate affects the benefits of the new firm. Lower revenue growth will decrease the expected present value of the project. This shifts the threshold curve in figure 3a upwards and makes a later market entry more desireable.

(iii) Path Level of Pre-start-up Expected Market Value and Revenue

Proposition 6 *A rising level of the pre-start-up expected market value $\tilde{P}_i(0)$ of the new product i will decrease the start-up gestation phase T^**

$$\frac{dT^*}{d\tilde{P}_i(0)} = \frac{1}{\underbrace{\frac{\lambda}{\lambda - 1}(r - \alpha)Ce^{(r-\delta)T} - \delta\tilde{P}_i(0)}_{<0}} < 0. \tag{19}$$

Proof. See Appendix 6

$\tilde{P}_i(0)$ indicates the level of the path of the potential pre-start-up market price (respectively profit) obtainable for product i when the business is conceptualised at $t = 0$. In case of high potential market values of the product the start-up phase will be shorter. This finding is intuitive. An increasing potential market value describes (other things equal including costs) that the product idea earns more in the market. E.g. $\tilde{P}_i(0)$ could indicate if the product i is regarded as a premium or standard product. The higher the expected market evaluation and hence the level of the expected price path, the more attractive is a quick market entry. This reaction is also illustrated in figure 3b. Since an increasing $\tilde{P}_i(0)$ leads to an upward shift of the expected market value curve, the threshold is reached more early and the market entry can be expected to be more rapid.

There is an alternative economic interpretation for $\tilde{P}_i(0)$. The variable $\tilde{P}_i(0)$ may identify the extra profits from entrepreneurial activities compared to the existing wage income. In this interpretation we would include opportunitiy costs from wage income in our considerations. If a person with a given start-up idea has high wage income, the level of extra profits would decline compared to someone with a low wage income. Hence decreasing the extra entrepreneurial profits above wage income would postphone entrepreneurial activities and reduce the willingness to start the business.

(iv) Start-Up Costs

Proposition 7 *With increasing start-up costs the expected time of market entry T^* will be postponed:*

$$\frac{dT^*}{dC} = \frac{-[e^{rT} - 1]}{\underbrace{\left(e^{rT} - \frac{\delta\,(\lambda - 1)}{(r - \alpha)\lambda}\frac{\tilde{P}_i(0)}{C}e^{\delta T}\right)rC}_{<0}} > 0. \tag{20}$$

Proof. See Appendix 7

As C denotes investment costs for conceptual planning and product development, or more general start-up preparation, $\frac{dT^*}{dC} > 0$ is the expected reaction. With increasing costs for product development the entrepreneur needs a compensation from the market. Therefore the required threshold curve shifts upwards in figure 3a. As long as the market would reward the outcome of the additional product improvement sufficiently enough, both curves would still intersect at a later time. Hence increasing costs for development would postpone the start-up but not push the idea out of the market. However, this is the most simple linear case discussed in the proposition. If the marginal market reward is not sufficient (upper dashed line in figure 3a) increasing costs could make the project unfeasible. The shift in the threshold cannot be matched by the market reward and we find no intersect. The start-up will not take place.

(v) Rewarding Rate for Product Quality Improvements:

Proposition 8 *An increase in the marginal market reward for product improvement is generally ambiguous. However, an increase in δ will tend to speed up market entry T^* if $\frac{\tilde{P}_i(0)}{C}$ becomes sufficiently large within the limits of conditions (15) and (25) hold,*

$$\frac{dT^*}{d\delta} = \frac{\overbrace{\ln(\frac{\lambda-1}{\lambda}\frac{1}{r-\alpha}\frac{\tilde{P}_i(0)}{\bar{C}})}^{<0 \text{ see (15)}}}{(\beta-\delta)^2} + \frac{1}{\beta-\delta}\frac{\overset{(-)}{\frac{\partial\lambda}{\partial\delta}}}{(\lambda-1)\,\lambda} > 0 \tag{21}$$

Proof. See Appendix 8.

Looking at conditon (21) the sign of the reaction depends on the relative importance of the two terms, i.e., we can identify two different effects for the gestation phase. On the one hand, as preparation and investments in product development and product improvement generate an increasing market reward, the virtual market value is approaching a threshold value for market entry more quickly [first term of (21)].

On the other hand, the threshold itself will be affected. Waiting time generates a higher reward and hence becomes more valueable [second term of (21)]. Depending on these two relative effects we obtain a positive or negative total effect. Further, as the threshold is - among others - determined by the evaluation of the waiting time, we can see how the real option approach affects the decision.

(vi) Interest Rate

Proposition 9 *An increase in the interest rate is generally ambiguous. However, an increase in the interest rate will reduce/increase education (decrease/increase T^*) if (26)/(27) holds*

$$\frac{dT^*}{dr} = \underbrace{\frac{1}{\beta - \delta}}_{(-)} \left[\underbrace{\frac{\overset{(+)}{\frac{\partial \lambda}{\partial r}}}{\lambda\left(\lambda - 1\right)} - \frac{1}{\left(r - \alpha\right)}}_{=:X \gtreqless 0} \right] \gtreqless 0 \tag{22}$$

Proof. See Appendix 9

As formally discussed in Appendix 9 the reaction is generally ambiguous. The two terms represent two effects in the decision. The first term represents that the effect of higher costs of borrowing, or opportunity costs, for the investement during the planning phase must be compensated by the positive price increase δ generated by the investment. The second term represents the effects generated by the change of the threshold condition.

4 Modelling Serial Entrepreneurship

In the previous section we derived a framework to determine the optimal timing of market entry in the case of a novice entrepreneur. In this section we extend this to the case of a serial entrepreneur. In the case of serial entrepreneurs we will have a sequence of start-ups. To illustrate the learning effects, we compare the difference between project n and $n + 1$. The timing between start-ups will depend on the properties of the next intended start-up as well as the fact that the personal characteristics of the entrepreneur will have been modified through the nature of his or her prior business experience. In this section we model how learning from past experience could affect the start-up timing decision.

As was mentioned, the crucial difference between novice and serial entrepreneurs is in their entrepreneurial ability, which could have been influenced by the nature of their prior business experience. In general most parameters are affected by the experience and the outcome of previous entrepreneurial ventures. The way how these experiences affect future activities may differ among entrepreneurs and may change with the environmental conditions. However, there are a number of ways in which an increasing number n of start-up experiences might affect the timing of start-up decision as modelled in the previous sections.

First, it could reduce preparation and start-up investment costs during the conception and gestation phase of the next business, i.e. $C_{n+1} = C(n) \quad \text{with} \quad dC/dn < 0$. With accumulated experience the entrepreneur knows better how to set up a firm. Information costs, costs on market observation and product placement would be reduced.

Second, it could improve the effectiveness of the entrepreneurs' start-up investments in product development and product quality during the gestation phase. Not just cost may be affected by learning, the entrepreneur also may know better how to develop and improve a product and gain additional market value, i.e. $\delta_{n+1} = \delta(n) \quad \text{with} \quad d\delta/dn > 0$.

Third, it could contribute to better risk management. A serial entrepreneur may be inclined to avoid activities that might lead to a high volatility of business revenues. As a result the project specific risk and hence volatility may decrease with increased entrepreneurial experience and ability i.e. $\sigma_{n+1} = \sigma_i(n) \quad \text{with} \quad d\sigma_i/dn < 0$.

Fourth, it may result in faster rates of business expansion once the new firm is started up by the serial entrepreneur. An experienced entrepreneur may know, how important a rapid market expansion might be and how to succeed in rapidly penetrating the target market segment. Hence, it can be expected that the rate of business expansion is a positive function of experience, i.e. $\alpha_{n+1} = \alpha(n) \quad \text{with} \quad d\alpha/dn > 0$.

Fifth, being a serial entrepreneur may change his or her ability to obtain finance. Whether it improves this ability may depend whether or not the entrepreneur has been previously successful or not. Thus the serial entrepreneur could either face a higher (as penalty for failure) or lower (reward for an expected lower personal risk) interest rates for externally raised loans, i.e. $r_{n+1} = r(n) \quad \text{with} \quad dr/dn \gtreqless 0$.

Sixth, having the benefit of previous experience may affect the level of the pre-start-up's expected net market value in various ways. For instance, the level of the path reflects to a certain extend the product niche the entrepreneur is targeting. Previous experiences may lead to a different choice of the market niche, for e.g. a switch from a standard product to a premium product. In this case the pre-start-up expected market value would rise, i.e. $\tilde{P}_i(0)_{n+1} = P_i(0,n) \quad \text{with} \quad d\tilde{P}_i(0)/dn > 0$.

However, we have not yet made a distinction in our model between market value (price), revenue and profits. We assumed no or fixed operating costs. A given output reflects a potential full time occupation the entrepreneur would devote to his or her business once it is started. Therefore the terms (net) market value, price, (net) revenue, and profit could be used interchangeable. If the entrepreneur regards a potential wage from an alternative risk free occupation as the minimum entrepreneurial income and hence as part of his cost, $\tilde{P}_i(0)$ is the net market value of his entrepreneurial activities. Therefore, building up human capital as in the case of serial entrepreneurship could increase wages of alternative occupations and decrease the net reward from entrepreneurial activity. In this case the entrepreneurial experiences from previous projects would result in a lower personal net market reward of the future entrepreneurial activity, and i.e. $\tilde{P}_i(0)_{n+1} = P_i(0,n) \quad \text{with} \quad d\tilde{P}_i(0)/dn < 0$. An illustrative example would be a potential entrepreneur who faces a sudden loss of his job. In this case he cannot calculate an alternative wage as the opportunity costs in the revenue and profit process. From his or her subjective perspective the net value of the start up project would increase. Thus the sudden loss of his or her job would shift the threshold downwards, triggering an earlier tart of entrepreneurial activity.

In figure 4 we illustrate the effects from previous entrepreneurial experiences for the duration of a business set up of a serial entrepreneur. Analyzing a serial entrepreneur allows to study the formation of entrepreneur-specific human capital. From a sequence of future entrepreneurial activities we can observe the learning process and the implications of learning by adjusting the strategy and the timing of entrepreneurial activities. To keep figure 4 simple we focus only on two potential earning effects, the lower costs for preparation and product development ($dC/dn < 0$), and the improved ability of appropriately addressing the customers needs resulting in a increasing level of the price path of the product value ($d\tilde{P}_i(0)/dn > 0$).

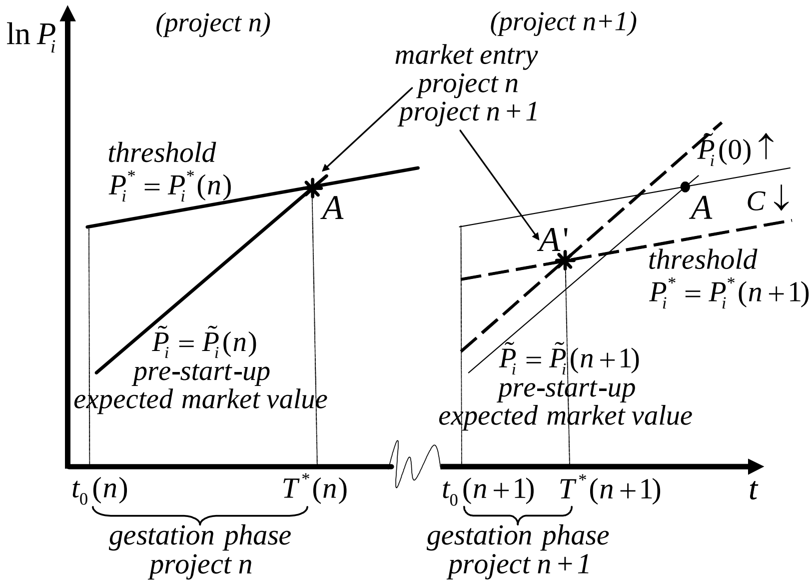

Figure 4: Learning, and timing of start-ups for a serial entrepreneur

When $t_0(n)$ is the time when the idea for project n appeared for the first time we can determine the market entry and hence the duration of the gestation phase for each project in a sequence of start-up activities. Using the hypothesis discussed above, figure 4 shows that for the two effects described, the duration of planning and conceptual activities will decrease. Generally the learning process can be expected to speed up market entry for a serial entrepreneur.

However, if an entrepreneur has a damaged reputation due to having failed in business before, he or she may have to pay higher interest rates when applying for loans. This penalty may counteract the positive learning effects and may eventually prevent the latent entrepreneur from starting up a new firm. More generally though, the results illustrate the importance of financial constraints on serial entrepreneurship. There may be many potential serial entrepreneurs with improved entrepreneurial ability in an economy, particularly in developing countries, who are prevented from benefitting from their experience by lack of access to finance.

Our model is flexible enough to study a number of interesting aspects of serial entrepreneurship - such as the effect of age on the start-up decision. However, due to space limitations we leave the illustration hereof, and the further eloboration of the model, for a future paper.

5 Concluding Remarks

We have shown in this paper that timing matters in the start-up decision of entrepreneurs, and how the optimal time to start a new firm can be determined, combining real option theory with the occupational choice theory of entrepreneurship. We have also shown how the phenomenon of serial entrepreneurship can be formally modelled in this context, since every novice entrepreneur is potentially a serial entrepreneur, but perhaps never finds the right time to start a next firm. Using the model derived in this paper we have highlighted that there is value for a potential entrepreneur to wait before starting a firm, and identified the importance of factors such as entrepreneurial ability, learning, innovation, access to finance and random market events in triggering or stalling market entry.

Our model has both theoretical and practical contributions. Theoretically, we integrated insights from *real option theory*, used especially in investment analysis with the theory on entrepreneurial market entry. Moreover, by distinguishing between the start-up timing decisions of novice and serial entrepreneurs we made a specific contribution to the study of serial entrepreneurship, given the current lack of formal theoretical models describing the behaviour of serial entrepreneurs.

Our model may have practical value for refining policies towards start-ups. This is because the impact of entrepreneurs on economic growth and development may depend on entrepreneurs starting their firms at the optimal time.

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6 Appendices

6.1 Appendix 1: Proof of Proposition 1 and Derivatives of λ

a) The value of the revenue stream is determined by

$$\begin{aligned} V^{gross} &= E\int_{T}^{\infty} e^{-r(t-T)}P_i dt \\ &= \int_{T}^{\infty} P_i e^{-r(t-T)} e^{\alpha(t-T)} \\ &= \left[\frac{1}{\alpha - r} e^{(\alpha-r)(t-T)} P_i\right]_T^{\infty} \\ &= \frac{P_i}{r-\alpha} \end{aligned}$$

b) For the option values F_i the Hamilton-Jacobi-Bellman equation for the Brownian motion of 2 holds:

$$rF = \frac{1}{dt}E(dF)$$

From Ito's Lemma we know:

$$\begin{aligned} dF &= (\frac{\partial F}{\partial t} + \delta\tilde{P}_i\frac{\partial F}{\partial\tilde{P}_i} + \frac{1}{2}\sigma^2\tilde{P}_i^2\frac{\partial F}{\partial\tilde{P}_i^2})dt + \sigma\tilde{P}_i\frac{\partial F}{\partial\tilde{P}_i}dW \\ \Rightarrow E(dF) &= (\frac{\partial F}{\partial t} + \delta\tilde{P}_i\frac{\partial F}{\partial\tilde{P}_i} + \frac{1}{2}\sigma^2\tilde{P}_i^2\frac{\partial F}{\partial\tilde{P}_i^2})dt \end{aligned}$$

because $E(dW) = 0$.
From the last two equations we obtain the following differential equation:

$$\underbrace{\frac{\partial F}{\partial t}}_{=0} + \delta\tilde{P}_i\frac{\partial F}{\partial\tilde{P}_i} + \frac{1}{2}\sigma^2\tilde{P}_i^2\frac{\partial F}{\partial\tilde{P}_i^2} - rF = 0$$

$$\Leftrightarrow \delta\tilde{P}_i\frac{\partial F}{\partial\tilde{P}_i} + \frac{1}{2}\sigma^2\tilde{P}_i^2\frac{\partial F}{\partial\tilde{P}_i^2} - rF = 0$$

This is a second-order homogenous ordinary differential equation with a free boundary.

c) A general solution to this differential equation will be

$$F = B\tilde{P}^{\lambda}.$$

$B\tilde{P}^{\lambda}$ solves the homogenous differential equation.

$$\begin{aligned}
\delta\tilde{P}_i B\lambda\tilde{P}_i^{\lambda-1} + \frac{1}{2}\sigma^2 B\tilde{P}_i^2\lambda(\lambda-1)\tilde{P}_i^{\lambda-2} - rB\tilde{P}_i^{\lambda} &= 0 \\
\delta B\lambda\tilde{P}_i^{\lambda} + \frac{1}{2}\sigma^2 B\lambda(\lambda-1)\tilde{P}_i^{\lambda} - rB\tilde{P}_i^{\lambda} &= 0 \\
\delta\lambda + \frac{1}{2}\sigma^2\lambda(\lambda-1) - r &= 0
\end{aligned}$$

$$\Leftrightarrow \quad \lambda = \frac{1}{2} - \frac{\delta}{\sigma^2} + \sqrt{(\frac{1}{2} - \frac{\delta}{\sigma^2})^2 + \frac{2r}{\sigma^2}} > 1 \qquad \text{see (11)}$$

with $\quad \delta \quad < \quad r \qquad$ see (12)

As $\tilde{P}_i$ goes to zero, F tends to approach 0. This implies that the negative root of the characteristic polynomial should have no influence on F as $\tilde{P}_i$ tends to zero.

Besides $\lambda > 1 \Leftrightarrow r > \delta$:

$$\begin{aligned}
\frac{1}{2} - \frac{\delta}{\sigma^2} + \sqrt{(\frac{1}{2} - \frac{\delta}{\sigma^2})^2 + \frac{2r}{\sigma^2}} &> 1 \\
\sqrt{(\frac{1}{2} - \frac{\delta}{\sigma^2})^2 + \frac{2r}{\sigma^2}} &> \frac{1}{2} + \frac{\delta}{\sigma^2} \\
(\frac{1}{2} - \frac{\delta}{\sigma^2})^2 + \frac{2r}{\sigma^2} &> \left(\frac{1}{2} + \frac{\delta}{\sigma^2}\right)^2 \\
-2\frac{\delta}{\sigma^2}\frac{1}{2} + \frac{2r}{\sigma^2} &> 2\frac{\delta}{\sigma^2}\frac{1}{2} \\
-\frac{\delta}{\sigma^2} + \frac{2r}{\sigma^2} &> \frac{\delta}{\sigma^2} \\
r &> \delta
\end{aligned}$$

For the derivatives of λ we get:

$$\begin{aligned}
\frac{d\lambda}{d\delta} &= -\frac{1}{\sigma^2} - \frac{2}{2}\left[\left(\frac{1}{2}-\frac{\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}\right]^{\frac{1}{2}-1}\left(\frac{1}{2}-\frac{\delta}{\sigma^2}\right)\frac{1}{\sigma^2} \\
&= -\frac{1}{\sigma^2}\left[1+\left[\left(\frac{1}{2}-\frac{\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}\right]^{-\frac{1}{2}}\left(\frac{1}{2}-\frac{\delta}{\sigma^2}\right)\right] < 0 \\
&= -\frac{\left[\left(\frac{1}{2}-\frac{\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}\right]^{-\frac{1}{2}}}{\sigma^2}\left[\left[\left(\frac{1}{2}-\frac{\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}\right]^{\frac{1}{2}} + \left(\frac{1}{2}-\frac{\delta}{\sigma^2}\right)\right] < 0 \\
&= -\frac{\left[\left(\frac{1}{2}-\frac{\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}\right]^{-\frac{1}{2}}}{\sigma^2}\lambda < 0
\end{aligned}$$

$$\frac{d\lambda}{dr} = \left[\left(\frac{1}{2}-\frac{\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}\right]^{-\frac{1}{2}}\frac{1}{\sigma^2} > 0$$

$$\begin{aligned}
\frac{d\lambda}{d\sigma} &= \frac{2\delta}{\sigma^3} + \frac{1}{2}\left[\left(\frac{1}{2}-\frac{\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}\right]^{-\frac{1}{2}}(2(\frac{1}{2}-\frac{\delta}{\sigma^2})\cdot\frac{2\delta}{\sigma^3} - \frac{4r}{\sigma^3}) \\
&= \frac{2\delta}{\sigma^3} + \left[\left(\frac{1}{2}-\frac{\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}\right]^{-\frac{1}{2}}((\frac{1}{2}-\frac{\delta}{\sigma^2})\cdot\frac{2\delta}{\sigma^3} - \frac{2r}{\sigma^3}) \\
&= \frac{2\delta}{\sigma^3}\left[1+\left[\left(\frac{1}{2}-\frac{\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}\right]^{-\frac{1}{2}}\left((\frac{1}{2}-\frac{\delta}{\sigma^2}) - \frac{r}{\delta}\right)\right] \\
&= \frac{2\delta\left[\left(\frac{1}{2}-\frac{\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}\right]^{-\frac{1}{2}}}{\sigma^3}\left[\left[\left(\frac{1}{2}-\frac{\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}\right]^{\frac{1}{2}} + \frac{1}{2} - \frac{\delta}{\sigma^2} - \frac{r}{\delta}\right] < 0
\end{aligned}$$

d) At the investment trigger point P_i^* the value of the option must equal the net value obtained by exercising it (value of the active project minus sunk cost of the investment). Hence the following must hold:

$$
\begin{aligned}
F(P_i^*) &= V^{gross}(P_i^*) - I(T). \\
&= \int_T^{\infty} P_i^* e^{-r(t-T)} e^{\alpha(t-T)} - \left[\int_0^T e^{r(T-t)} C dt + \bar{C}\right] \\
&= \left[\frac{1}{\alpha - r} e^{(\alpha-r)(t-T)} P_i^*\right]_T^{\infty} - \left[\left[-\frac{C}{r} e^{r(T-t)}\right]_0^T + \bar{C}\right] \\
&= 0 - \frac{P_i^*}{\alpha - r} e^{(\alpha-r)(T-T)} - \left(-\frac{C}{r} + \frac{C}{r} e^{rT} + \bar{C}\right) \\
&= \frac{P_i^*}{r - \alpha} - \frac{C}{r}\left(e^{rT} - 1\right) - \bar{C}
\end{aligned}
$$

$$
B(P_i^*)^{\lambda} = \frac{P_i^*}{r - \alpha} - \frac{C}{r}\left(e^{rT} - 1\right) - \bar{C}
$$

Besides for $I(T) > 0$ we have to assume that $\bar{C} > \frac{C}{r}$.

The smooth-pasting condition requires that the two value functions meet tangentially:

$$
(F(P_i^*))' = (V^{gross}(P_i^*))'
$$

$$
\Leftrightarrow B\lambda(P_i^*)^{\lambda-1} = \frac{1}{r - \alpha}
$$

This implies

$$
B(P_i^*)^{\lambda} = \frac{P_i^*}{(r - \alpha)\lambda}
$$

Now we compute the threshold P_i^*:

$$
\frac{P_i^*}{r - \alpha} - \frac{C}{r}\left(e^{rT} - 1\right) - \bar{C} = \frac{P_i^*}{(r - \alpha)\lambda}
$$

$$
\Leftrightarrow \frac{P_i^*\lambda - P_i^*}{(r - \alpha)\lambda} = \frac{C}{r}\left(e^{rT} - 1\right) + \bar{C}
$$

$$
\Leftrightarrow P_i^*(\lambda - 1) = (r - \alpha)\lambda\left[\frac{C}{r}\left(e^{rT} - 1\right) + \bar{C}\right]
$$

$$
\Leftrightarrow P_i^*(T) = \frac{\lambda}{\lambda - 1}(r - \alpha)\left[\frac{C}{r}\left(e^{rT} - 1\right) + \bar{C}\right] = \frac{\lambda}{\lambda - 1}(r - \alpha) I(T)
$$

e) With $\ln I(T)$ being convex and hence $\ln P_i^*(T)$ being a convex function in T.:

$$
\frac{\partial \ln I}{\partial T} = \frac{C e^{rT}}{\frac{C}{r}\left(e^{rT} - 1\right) + \bar{C}} > 0,
$$

$$
\begin{aligned}
\frac{\partial^2 \ln I}{\partial T^2} &= \frac{Cre^{rT}(\frac{C}{r}\left(e^{rT}-1\right)+\bar{C})-C^2e^{2rT}}{(\frac{C}{r}\left(e^{rT}-1\right)+\bar{C})^2} > 0 \\
&= \frac{Cre^{rT}\frac{C}{r}e^{rT}-\frac{C}{r}Cre^{rT}+Cre^{rT}\bar{C})-C^2e^{2rT}}{(\frac{C}{r}\left(e^{rT}-1\right)+\bar{C})^2} \\
&= \frac{-\frac{C}{r}Cre^{rT}+Cre^{rT}\bar{C}}{(\frac{C}{r}\left(e^{rT}-1\right)+\bar{C})^2} \\
&= \frac{Ce^{rT}\left(r\bar{C}-C\right)}{(\frac{C}{r}\left(e^{rT}-1\right)+\bar{C})^2} > 0 \text{ (convex) as we assume condition (16)} \qquad (23)
\end{aligned}
$$

$$
\begin{aligned}
\lim_{T\to\infty}\frac{\partial \ln I}{\partial T} &= \lim_{T\to\infty}\frac{Ce^{rT}}{\frac{C}{r}\left(e^{rT}-1\right)+\bar{C}} = \lim_{T\to\infty}\frac{re^{rT}}{\left[e^{rT}-1+\frac{r\bar{C}}{C}\right]} \\
&= \lim_{T\to\infty}\frac{re^{rT}}{\left[1+\frac{-1+\frac{r\bar{C}}{C}}{e^{rT}}\right]e^{rT}} = r
\end{aligned}
$$

6.2 Appendix 2: Deriving T and Proof of Proposition 2

a) Development of the initial income level value: The development of the pre-start-up market value is determined by

$$d\tilde{P}_i = \delta\tilde{P}_i + \sigma\tilde{P}_i dW$$

We put $g(x) = \log x$ to get the Ito formula for $\log \tilde{P}_i(t)$:

$$
\begin{aligned}
d(\log \tilde{P}_i(t)) &= (\frac{1}{\tilde{P}_i(t)}\delta\tilde{P}_i(t) + \frac{1}{2}(-\frac{1}{\tilde{P}_i(t)^2})\tilde{P}_i{}^2\sigma^2)dt + \frac{1}{\tilde{P}_i(t)}\sigma\tilde{P}_i(t)dW \\
&= (\delta - \frac{1}{2}\sigma^2)dt + \sigma dW
\end{aligned}
$$

We obtain after integration

$$
\begin{aligned}
\log \tilde{P}_i(T) - \log \tilde{P}_i(0) &= \int_0^T (\delta - \frac{1}{2}\sigma^2)dt + \int_0^T \sigma dW \\
&\Leftrightarrow \log \tilde{P}_i(T) = \log \tilde{P}_i(0) + (\delta - \frac{1}{2}\sigma^2)T + \sigma W(T), \text{ and hence} \\
\tilde{P}_i(T) &= \tilde{P}_i(0)e^{((\delta-\frac{1}{2}\sigma^2)T+\sigma W(T))} \quad \text{and hence} \\
E\tilde{P}_i(T) &= \tilde{P}_i(0)e^{\delta T}.
\end{aligned}
$$

$$\frac{\partial E\tilde{P}_i(T)}{\partial T} = \delta\tilde{P}_i(0)e^{\delta T}$$

and $\ln E\tilde{P}_i(T)$ is a linear function in T:

$$\ln E\tilde{P}_i(T) = \ln \tilde{P}_i(0) + \delta T$$

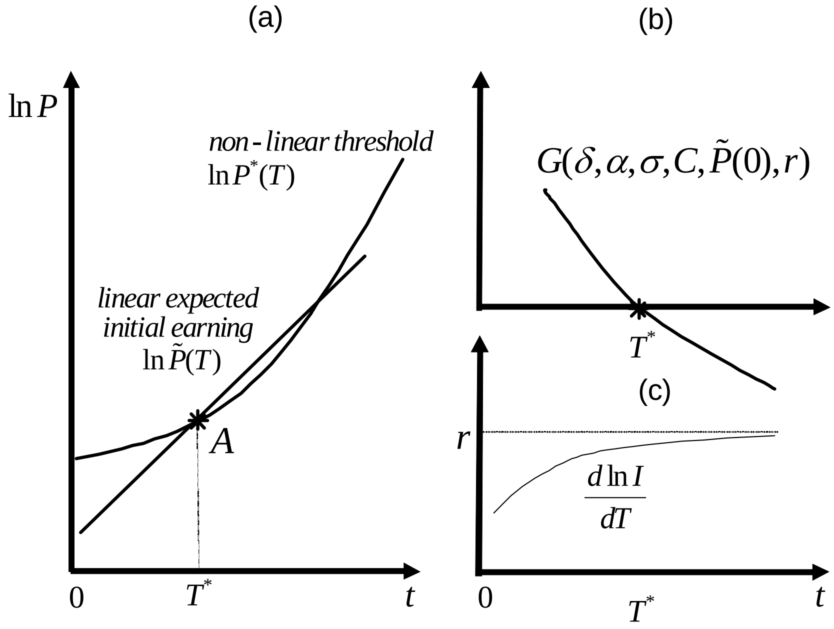

Figure 5:

6.3 Appendix 3: Existence of a solution for the expected time T^* of market entry, and determination of T^* as an implicit function/Proof of Proposition 3:

In general we look for conditions described in figure 5. The threshold starts above the initial income curve. For positive T the threshold will have an unique intersection with the initial value curve from below at A. Hence at the time of expected market entry denoted by T^* $G = P_i^*(T) - E\tilde{P}_i(T) = 0$ and the $G-$curve has a negative slope $\frac{dG}{dT} < 0$.

Further, at T^* the threshold $P_i^*(T = 0)$ must start above the expected initial income $E\tilde{P}_i(T = 0)$, and $G > 0$ during the pre-market entry period $(0 < t < T^*)$. Otherwise the market entry would have been taken place.

6.3.1 Negative slope of G

$$\frac{\partial G}{\partial T^*} = \frac{\lambda}{\lambda-1}(r-\alpha)Ce^{rT^*} - \delta\tilde{P}_i(0)e^{\delta T^*} < 0 \tag{24}$$

$$\Leftrightarrow \tilde{P}_i(0) > \frac{\lambda}{\lambda-1}\frac{(r-\alpha)}{\delta}Ce^{(r-\delta)T^*}$$

$$\Leftrightarrow \frac{\lambda}{\lambda-1}(r-\alpha)\left[\frac{C}{r}e^{(r-\delta)T^*} - (\frac{C}{r}-\bar{C})e^{-\delta T^*}\right] > \frac{\lambda}{\lambda-1}\frac{(r-\alpha)}{\delta}Ce^{(r-\delta)T^*}$$

$$\Leftrightarrow \frac{C}{r}e^{(r-\delta)T^*} - (\frac{C}{r}-\bar{C})e^{-\delta T^*} > \frac{C}{\delta}e^{(r-\delta)T^*}$$

$$\Leftrightarrow \frac{C}{r} + (-\frac{C}{r}+\bar{C})e^{-rT^*} > \frac{C}{\delta}$$

$$\Leftrightarrow (-\frac{C}{r}+\bar{C})e^{-rT^*} > \frac{C}{\delta} - \frac{C}{r}$$

$$\Leftrightarrow e^{-rT^*} > \frac{\frac{C}{\delta}-\frac{C}{r}}{-\frac{C}{r}+\bar{C}}$$

$$\Leftrightarrow -rT^* > \ln\left(\frac{\frac{C}{\delta}-\frac{C}{r}}{-\frac{C}{r}+\bar{C}}\right) \Leftrightarrow T^* < \frac{-1}{r}\ln\left(\frac{\frac{C}{\delta}-\frac{C}{r}}{-\frac{C}{r}+\bar{C}}\right)$$

$$\text{and } \frac{\frac{C}{\delta}-\frac{C}{r}}{-\frac{C}{r}+\bar{C}} < 1 \Leftrightarrow \frac{C}{\delta} - \frac{C}{r} < -\frac{C}{r} + \bar{C}$$

$$\Leftrightarrow \frac{C}{\delta} < \bar{C} \Leftrightarrow C < \delta\bar{C} < r\bar{C} \quad see \quad 16$$

Before market entry the initial income curve must grow faster than the threshold curve. Only for a negative slope G can approach and eventually reach zero. $\frac{\partial G}{\partial T} < 0$ is fulfilled if condition $\bar{C} > \frac{C}{r}$ (condition 16)

6.3.2 Existence of an intersect of $P_i^*(T^*)$ and $E\tilde{P}_i(T^*)$ for positive T^*

a) As the function $\ln P_i^*(T)$ is convex if condition (16) holds (see 23) and the function $\ln E\tilde{P}_i(T)$ is linear, there are at most two intersections. We are interested only in intersections at $T > 0$. An intersection for positive values of both functions exsits if condition (15) and (16) holds and $G = 0$ for positive values of T^*.

$$\begin{aligned}
G &= P_i^*(T^*) - E\tilde{P}_i(T^*) = 0 \\
&= \frac{\lambda}{\lambda-1}(r-\alpha)\left[\frac{C}{r}(e^{rT^*}-1)+\bar{C}\right] - \tilde{P}_i(0)e^{\delta T^*} = 0 \\
&\Leftrightarrow \ln\left[\frac{\lambda}{\lambda-1}(r-\alpha)\left[\frac{C}{r}(e^{rT^*}-1)+\bar{C}\right]\right] = \ln\tilde{P}_i(0) + \delta T^* \\
&\Leftrightarrow \ln\left[\frac{\lambda}{\lambda-1}(r-\alpha)\left[\frac{C}{r}(e^{rT^*}-1)+\bar{C}\right]\right] - \ln\tilde{P}_i(0) = \delta T^* \\
&\Leftrightarrow \frac{1}{\delta}\ln\left[\frac{\frac{\lambda}{\lambda-1}(r-\alpha)\left[\frac{C}{r}(e^{rT^*}-1)+\bar{C}\right]}{\tilde{P}_i(0)}\right] = T^* \overset{!}{>} 0
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \frac{\lambda}{\lambda-1}(r-\alpha)\left[\frac{C}{r}(e^{rT^*}-1)+\bar{C}\right] > \tilde{P}_i(0) \\
&\Leftrightarrow \frac{\lambda}{\lambda-1}(r-\alpha)\frac{C}{r}e^{rT^*} > \tilde{P}_i(0) + \frac{\lambda}{\lambda-1}(r-\alpha)\left(\frac{C}{r}-\bar{C}\right) \\
&\Leftrightarrow e^{rT^*} > \frac{\lambda-1}{\lambda}\frac{1}{(r-\alpha)}\frac{r}{C}\tilde{P}_i(0) + 1 - \bar{C}\frac{r}{C} \\
&\Leftrightarrow T^* > \frac{1}{r}\ln\Bigg[\underbrace{\frac{\lambda-1}{\lambda}\frac{1}{(r-\alpha)}\frac{r}{C}\tilde{P}_i(0) + 1 - \frac{r}{C}\bar{C}}_{<1}\Bigg] \text{see (15)} \\
&\qquad\Leftrightarrow \tfrac{r}{C}\bar{C} > \tfrac{\lambda-1}{\lambda}\tfrac{1}{(r-\alpha)}\tfrac{r}{C}\tilde{P}_i(0) \\
&\qquad\Leftrightarrow \bar{C} > \tfrac{\lambda-1}{\lambda}\tfrac{1}{(r-\alpha)}\tilde{P}_i(0)
\end{aligned}$$

The last inequality is a condition for the axis intercepts of $P_i^*$and $E\tilde{P}_i$. It guarantees that $E\tilde{P}_i$ has a lower value in $T = 0$ than P_i^*.

$$\begin{aligned}
P_i^*(0) &> E\tilde{P}_i(0) \\
&\Rightarrow \bar{C} > \frac{\lambda-1}{\lambda}\frac{1}{(r-\alpha)}\tilde{P}_i(0)
\end{aligned}$$

b) Further, in figure 5 the condition for an intersection and a negative slope have to hold simultaneously at T^*. We need to show that there is a T^* were both $\frac{dG}{dT} < 0$ and $G = 0$ hold. That is, we can find a minimum level for $\tilde{P}_i(0)$

in order to ensure an intersection and a negative slope:

$$\underbrace{\frac{1}{\delta}\ln\left[\frac{\frac{\lambda}{\lambda-1}(r-\alpha)\left[\frac{C}{r}(e^{rT^*}-1)+\bar{C}\right]}{\tilde{P}_i(0)}\right]}_{\text{follows from } G=0} \quad = \quad T^* \quad < \underbrace{\frac{-1}{r}\ln\left(\frac{\frac{C}{\delta}-\frac{C}{r}}{-\frac{C}{r}+\bar{C}}\right)}_{\text{follows from the slope condition}}$$

$$\frac{1}{\delta}\ln\left[\frac{\lambda}{\lambda-1}(r-\alpha)\left[\frac{C}{r}(e^{rT^*}-1)+\bar{C}\right]\right]-\frac{1}{\delta}\ln\tilde{P}_i(0) \quad < \quad \frac{-1}{r}\ln\left(\frac{\frac{C}{\delta}-\frac{C}{r}}{-\frac{C}{r}+\bar{C}}\right)$$

$$\ln\left[\frac{\lambda}{\lambda-1}(r-\alpha)\left[\frac{C}{r}(e^{rT^*}-1)+\bar{C}\right]\right]+\frac{\delta}{r}\ln\left(\frac{\frac{C}{\delta}-\frac{C}{r}}{-\frac{C}{r}+\bar{C}}\right) \quad < \quad \ln\tilde{P}_i(0)$$

$$\ln\left[\frac{\lambda}{\lambda-1}(r-\alpha)\left[\frac{C}{r}(e^{rT^*}-1)+\bar{C}\right]\right]+\underbrace{\frac{\delta}{r}\ln\left(\frac{\frac{C}{\delta}-\frac{C}{r}}{-\frac{C}{r}+\bar{C}}\right)}_{c<0} \quad < \quad \ln\tilde{P}_i(0)$$

$$\left[\frac{\lambda}{\lambda-1}(r-\alpha)\left[\frac{C}{r}(e^{rT^*}-1)+\bar{C}\right]\right]e^c \quad < \quad \tilde{P}_i(0)$$

$$\frac{\lambda}{\lambda-1}(r-\alpha)\frac{C}{r}e^{rT^*}e^c-\frac{\lambda}{\lambda-1}(r-\alpha)\frac{C}{r}e^c+\frac{\lambda}{\lambda-1}(r-\alpha)\bar{C}e^c \quad < \quad \tilde{P}_i(0)$$

$$\begin{aligned}
\frac{\lambda}{\lambda-1}(r-\alpha)\frac{C}{r}e^{rT^*}e^c \quad &< \quad \tilde{P}_i(0)+\frac{\lambda}{\lambda-1}(r-\alpha)\frac{C}{r}e^c-\frac{\lambda}{\lambda-1}(r-\alpha)\bar{C}e^c \\
e^{rT^*}e^c \quad &< \quad \tilde{P}_i(0)\frac{r}{C}\frac{1}{(r-\alpha)}\frac{\lambda-1}{\lambda}+e^c-\frac{r\bar{C}}{C}e^c \\
e^{rT^*} \quad &< \quad \tilde{P}_i(0)\frac{r}{C}\frac{1}{(r-\alpha)}\frac{\lambda-1}{\lambda}e^{-c}+1-\frac{r\bar{C}}{C} \\
T^* \quad &< \quad \frac{1}{r}\ln\left(\tilde{P}_i(0)\frac{r}{C}\frac{1}{(r-\alpha)}\frac{\lambda-1}{\lambda}e^{-c}+1-\frac{r\bar{C}}{C}\right)
\end{aligned}$$

$$\begin{aligned}
\tilde{P}_i(0)\frac{r}{C}\frac{1}{(r-\alpha)}\frac{\lambda-1}{\lambda}e^{-c}+1-\frac{r\bar{C}}{C} \quad &> \quad 1 \\
\tilde{P}_i(0) \quad &> \quad \bar{C}\frac{\lambda}{\lambda-1}(r-\alpha)e^c
\end{aligned}$$

$$\tilde{P}_i(0) > \bar{C}\frac{\lambda}{\lambda-1}(r-\alpha)\underbrace{\left(\frac{\frac{C}{\delta}-\frac{C}{r}}{-\frac{C}{r}+\bar{C}}\right)^{\frac{\delta}{r}}}_{<1}$$

Finally, $\tilde{P}_i(0)$ has to lie in the open interval $\left(\bar{C}\frac{\lambda}{\lambda-1}(r-\alpha)\ ,\ \bar{C}\frac{\lambda}{\lambda-1}(r-\alpha)\left(\frac{\frac{C}{\delta}-\frac{C}{r}}{-\frac{C}{r}+\bar{C}}\right)^{\frac{\delta}{r}}\right)$.

6.3.3 c) T^* as implicit function of various variables: Proof of propositon 3

Proof of Proposition 3: (i) condition , (15) hold,

(ii) the derivative $\frac{\partial G}{\partial T}(\alpha, r, \sigma, T^*, C, \tilde{P}_i(0), \delta, \bar{C})$ is negative (see condition (24)) for each vector $(\alpha, r, \sigma, T^*, C, \tilde{P}_i(0), \delta, \bar{C})$ and

(iii) the partial derivatives of G by of $\alpha, \sigma, C, \tilde{P}_i(0), \delta, \bar{C}$ and r are continuous (vide infra), we can apply the implicit function theorem. Hence for a marginal environment of any vector $(\alpha, r, \sigma, T^*, C, \tilde{P}_i(0), \delta, \bar{C})$, T^* is an implicit function of of $\alpha, \sigma, C, \tilde{P}_i(0), \delta, \bar{C}$ and r. q.e.d.

$$T^* = T^*(\alpha, \sigma, C, \tilde{P}_i(0), \delta\ , r, \bar{C})$$

6.3.4 d) Curve properties of $V = V^{gross} - I$ (Net Current Value)

$$\begin{aligned} V &= \frac{\tilde{P}_i(0)e^{\delta T}}{r-\alpha} - \frac{C}{r}\left(e^{rT} - 1\right) - \bar{C} \\ &\Rightarrow\ (V^{gross} - I)(0) = \frac{\tilde{P}_i(0)}{r-\alpha} - \bar{C} \end{aligned}$$

$$\frac{d(V)}{dT} = \frac{\delta \tilde{P}_i(0)e^{\delta T}}{r-\alpha} - Ce^{rT}$$

Maximum of the curve:

$$\begin{aligned} 0 &= \frac{d(V)}{dT} = \frac{\delta \tilde{P}_i(0)e^{\delta T}}{r-\alpha} - Ce^{rT} \Rightarrow \ln\left[\frac{\delta \tilde{P}_i(0)}{r-\alpha}\right] + \delta T = \ln C + rT \\ &\Leftrightarrow\ T = \frac{1}{r-\delta}\ln\left[\frac{\tilde{P}_i(0)}{r-\alpha}\frac{\delta}{C}\right] \end{aligned}$$

$$\begin{aligned} \frac{d^2(V)}{dT^2} &= \frac{\delta^2 \tilde{P}_i(0)e^{\delta T}}{r-\alpha} - rCe^{rT} < 0 \\ &\Leftrightarrow\ rCe^{rT} > \frac{\delta^2 \tilde{P}_i(0)e^{\delta T}}{r-\alpha} \\ &\Leftrightarrow\ \ln(rC) + rT > \ln(\frac{\delta^2 \tilde{P}_i(0)}{r-\alpha}) + \delta T \\ &\Leftrightarrow\ T > \frac{1}{r-\delta}\ln(\frac{\delta^2 \tilde{P}_i(0)}{r-\alpha}\frac{1}{rC}) \\ \text{for } T &= \frac{1}{r-\delta}\ln\left[\frac{\tilde{P}_i(0)}{r-\alpha}\frac{\delta}{C}\right] \text{ we get} \\ \frac{\delta}{r} &> 1 \end{aligned}$$

$$\frac{\partial V}{\partial C} = -\frac{1}{r} e^{rT} < 0$$

6.4 Appendix 4: Proof of Proposition 4

To apply comparative statics for the implicit function $T^* = T^*(\alpha, \sigma, C, \tilde{P}_i(0), \delta\ r)$ we need to consider

$$\frac{\partial G}{\partial T} = \frac{\lambda}{\lambda - 1}(r-\alpha) C e^{rT} - \delta \tilde{P}_i(0) e^{\delta T} \gtreqless 0$$

Since we are only interested in values of T^* described by point A in figure 5 conditions (15) and $\frac{\partial G}{\partial T^*} < 0$ (16). Then at T^* we obtain:

$$\begin{aligned}
\frac{dT^*}{d\sigma} &= -\frac{\frac{dG}{d\sigma}}{\frac{\partial G}{\partial T}} = \frac{-1}{\frac{\lambda}{\lambda-1}(r-\alpha)Ce^{rT} - \delta\tilde{P}_i(0)e^{\delta T}}\left[\frac{-\frac{\partial\lambda}{\partial\sigma}}{(\lambda-1)^2}\{(r-\alpha)\frac{C}{r}e^{rT} - (r-\alpha)\frac{C}{r} + \bar{C}(r-\alpha)\}\right] \\
&= \frac{1}{\frac{\lambda}{\lambda-1}(r-\alpha)Ce^{rT} - \delta\tilde{P}_i(0)e^{\delta T}} \frac{\overset{(-)}{\frac{\partial\lambda}{\partial\sigma}}}{(\lambda-1)^2}(r-\alpha)\left(\frac{C}{r}\left(e^{rT}-1\right) + \bar{C}\right) > 0 \\
&= \frac{1}{\left(e^{rT} - \frac{\delta(\lambda-1)}{(r-\alpha)\lambda}\frac{\tilde{P}_i(0)}{C}e^{\delta T}\right)\frac{(r-\alpha)\lambda}{\lambda-1}C} \frac{\overset{(-)}{\frac{\partial\lambda}{\partial\sigma}}}{(\lambda-1)^2}(r-\alpha)\frac{C}{r}\left(e^{rT} - 1 + \frac{\bar{C}r}{C}\right) > 0 \\
&= \underbrace{\frac{\left(e^{rT} - 1 + \frac{\bar{C}r}{C}\right)}{\left(e^{rT} - \frac{\delta(\lambda-1)}{(r-\alpha)\lambda}\frac{\tilde{P}_i(0)}{C}e^{\delta T}\right)}}_{<0 \text{ see } (24 \text{ and } 16)} \frac{\overset{(-)}{\frac{\partial\lambda}{\partial\sigma}}}{\lambda(\lambda-1)r} > 0
\end{aligned}$$

6.5 Appendix 5: Proof of Proposition 5

$$\begin{aligned}
\frac{dT^*}{d\alpha} &= -\frac{\frac{dG}{d\alpha}}{\frac{\partial G}{\partial T}} = \frac{-1}{\frac{\lambda}{\lambda-1}(r-\alpha)Ce^{rT} - \delta\tilde{P}_i(0)e^{\delta T}}\left[-\frac{\lambda}{\lambda-1}\frac{C}{r}e^{rT} + \frac{\lambda}{\lambda-1}\frac{C}{r} - \bar{C}\frac{\lambda}{\lambda-1}\right] \\
&= \frac{1}{\left(e^{rT} - \frac{\delta(\lambda-1)}{(r-\alpha)\lambda}\frac{\tilde{P}_i(0)}{C}e^{\delta T}\right)\frac{\lambda(r-\alpha)}{\lambda-1}C} \frac{\lambda}{\lambda-1}\frac{C}{r}\left[e^{rT} - 1 + \frac{\bar{C}r}{C}\right] < 0 \\
&= \underbrace{\frac{\left[e^{rT} - 1 + \frac{\bar{C}r}{C}\right]}{\left(e^{rT} - \frac{\delta(\lambda-1)}{(r-\alpha)\lambda}\frac{\tilde{P}_i(0)}{C}e^{\delta T}\right)}}_{<0 \text{ see } (24 \text{ and } 16)} \frac{1}{(r-\alpha)r} < 0
\end{aligned}$$

6.6 Appendix 6: Proof of Proposition 6

$$\begin{aligned}\frac{dT^*}{d\tilde{P}_i(0)} &= -\frac{\frac{dG}{d\tilde{P}_i(0)}}{\frac{\partial G}{\partial T}} = \frac{-1}{\frac{\lambda}{\lambda-1}(r-\alpha)Ce^{rT} - \delta\tilde{P}_i(0)e^{\delta T}}[-e^{\delta T}] \\ &= \frac{1}{\underbrace{\frac{\lambda}{\lambda-1}(r-\alpha)Ce^{(r-\delta)T} - \delta\tilde{P}_i(0)}_{<0 \text{ see (24 and 16)}}} < 0\end{aligned}$$

6.7 Appendix 7: Proof of Proposition 7

$$\begin{aligned}\frac{dT^*}{dC} &= -\frac{\frac{dG}{dC}}{\frac{\partial G}{\partial T}} = \frac{-1}{\frac{\lambda}{\lambda-1}(r-\alpha)Ce^{rT} - \delta\tilde{P}_i(0)e^{\delta T}}[\frac{\lambda}{\lambda-1}(r-\alpha)\frac{1}{r}e^{rT} - \frac{\lambda}{\lambda-1}(r-\alpha)\frac{1}{r}] \\ &= \frac{-1}{\frac{\lambda}{\lambda-1}(r-\alpha)Ce^{rT} - \delta\tilde{P}_i(0)e^{\delta T}}\frac{\lambda}{\lambda-1}(r-\alpha)\frac{1}{r}[e^{rT}-1] \\ &= \frac{-1}{\left(e^{rT} - \frac{\delta(\lambda-1)}{(r-\alpha)\lambda}\frac{\tilde{P}_i(0)}{C}e^{\delta T}\right)\frac{\lambda(r-\alpha)}{\lambda-1}C}\frac{\lambda}{\lambda-1}(r-\alpha)\frac{1}{r}[e^{rT}-1] \\ &= \frac{-[e^{rT}-1]}{\underbrace{\left(e^{rT} - \frac{\delta(\lambda-1)}{(r-\alpha)\lambda}\frac{\tilde{P}_i(0)}{C}e^{\delta T}\right)}_{<0 \text{ see (24 and 16)}} rC} > 0\end{aligned}$$

6.8 Appendix 8: Proof of Proposition 8

For the derivative with respect to δ and r we need an approximation of $I(T)$ to examine the sign. We approximate $I(T)$ for the time range between 0 and the point T^* by a log linear function with the parameter β denoting the average growth rate of total accumulated costs between 0 and T^*. Economically this simplification describes an approximation where total costs are payable only at the end of the education period. The non log linear path of cost accumulation is proximated as a continous geometric growth process. Therefore we introduce a parameter β which determines the average growth rate of $I(T)$.

$$\bar{C}e^{\beta T} \approx \frac{C}{r}\left(e^{rT}-1\right) + \bar{C}$$

Note that we approximate a non-linear function (I) with a non-linear growth rate through a log-linear function which has the same unique positive intersection with the logarithmized income threshold curve. We consider the shape of $\ln I(T)$, which is convex as shown in appendix 1 condition (23).

$$
\begin{aligned}
\frac{\partial \ln I}{\partial T} &= \frac{Ce^{rT}}{\frac{C}{r}\left(e^{rT}-1\right)+\bar{C}} > 0, \\
&= \frac{Ce^{rT}\left(r\bar{C}-C\right)}{\left(\frac{C}{r}\left(e^{rT}-1\right)+\bar{C}\right)^2} > 0 \quad \text{since (16)} \\
&= \lim_{T\to\infty} \frac{re^{rT}}{\left[1+\frac{-1+\frac{r\bar{C}}{C}}{e^{rT}}\right]e^{rT}} = r
\end{aligned}
$$

and obviously

$$
\ln\left[\bar{C}e^{\beta T}\right]_{T=0} = \ln\bar{C} = \ln I(0).
$$

As both curves intersect in $T = T^*$ we can determine a β that satisfies the condition $\bar{C}e^{\beta T} \approx \frac{C}{r}\left(e^{rT}-1\right)+\bar{C}$:

$$
\begin{aligned}
\bar{C}e^{\beta T^*} &\approx \frac{C}{r}\left(e^{rT^*}-1\right)+\bar{C} \\
\beta T^* &= \ln\left[\frac{C}{r\bar{C}}\left(e^{rT^*}-1\right)+1\right] \\
\beta &= \frac{\ln\left[\frac{C}{r\bar{C}}\left(e^{rT^*}-1\right)+1\right]}{T^*}
\end{aligned}
$$

As $P_i^*(0) > E\tilde{P}_i(0)$ and $ln(\bar{C}e^{\beta T^*})$ and $lnI(T)$ start at the same point, the corresponding condition for the approximation to (24) is

$$
\delta - \beta > 0 \tag{25}
$$

Plugging the above approximation into the threshold we can explicitly determine T^* :

$$
\begin{aligned}
\frac{\lambda}{\lambda-1}(r-\alpha)\bar{C}e^{\beta T^*} - \tilde{P}_i(0)e^{\delta T^*} &= 0 \\
&\Leftrightarrow T^* = \ln\left(\frac{\lambda-1}{\lambda}\frac{1}{r-\alpha}\frac{\tilde{P}_i(0)}{\bar{C}}\right)\frac{1}{\beta-\delta} \\
&> 0, \text{ for } \frac{\lambda-1}{\lambda}\frac{1}{r-\alpha} < \frac{\bar{C}}{\tilde{P}_i(0)}. \quad \text{see (15)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{dT^*}{d\delta} &= \frac{1}{(\beta-\delta)^2}\ln\left(\frac{\lambda-1}{\lambda}\frac{1}{r-\alpha}\frac{\tilde{P}_i(0)}{\bar{C}}\right) + \frac{1}{\beta-\delta}\frac{\lambda}{\lambda-1}(r-\alpha)\frac{\bar{C}}{\tilde{P}_i(0)}\frac{\frac{\partial\lambda}{\partial\delta}}{\lambda^2}\frac{1}{r-\alpha}\frac{\tilde{P}_i(0)}{\bar{C}} \\
&= \frac{1}{(\beta-\delta)^2}\overbrace{\ln\left(\frac{\lambda-1}{\lambda}\frac{1}{r-\alpha}\frac{\tilde{P}_i(0)}{\bar{C}}\right)}^{<0 \text{ see (15)}} + \frac{1}{\beta-\delta}\frac{1}{\lambda-1}\frac{\overset{(-)}{\frac{\partial\lambda}{\partial\delta}}}{\lambda} > 0
\end{aligned}
$$

Similar to the derivative of T^* with respect to r we have to examine under which condition which summand prevails. Here we assume that the effect of the option value is dominant.

6.9 Appendix 9: Proof of Proposition 9

$$\frac{dT^*}{dr} = \frac{1}{\beta-\delta}\frac{\lambda}{\lambda-1}(r-\alpha)\frac{\bar{C}}{\tilde{P}_i(0)}\left[\frac{\frac{\partial\lambda}{\partial r}}{\lambda^2}\frac{1}{r-\alpha}\frac{\tilde{P}_i(0)}{\bar{C}} - \frac{\lambda-1}{\lambda}\frac{1}{(r-\alpha)^2}\frac{\tilde{P}_i(0)}{\bar{C}}\right]$$

$$\begin{aligned}\frac{dT^*}{dr} &= \frac{1}{\beta-\delta}\frac{\lambda}{\lambda-1}(r-\alpha)\frac{\bar{C}}{\tilde{P}_i(0)}\left[\frac{\frac{\partial\lambda}{\partial r}}{\lambda^2}\frac{1}{r-\alpha}\frac{\tilde{P}_i(0)}{\bar{C}} - \frac{\lambda-1}{\lambda}\frac{1}{(r-\alpha)^2}\frac{\tilde{P}_i(0)}{\bar{C}}\right] \\ &= \underbrace{\frac{1}{\beta-\delta}}_{(-)}\left[\underbrace{\frac{\overset{(+)}{\frac{\partial\lambda}{\partial r}}}{\lambda(\lambda-1)} - \frac{1}{(r-\alpha)}}_{=:X}\right]\end{aligned}$$

To find out if $X \gtreqless 0$ we need to follow three steps:

1) Assume a sufficient condition for $X > 0$ and $X < 0$: From our knowledge of the system we assume a sufficient condition for an unambigious sign. It is supposed that

$$\tfrac{3}{8}\sigma^2 + \tfrac{3}{2}\delta > r \quad \text{implies } X > 0. \tag{26}$$

$$\tfrac{3}{8}\sigma^2 + \tfrac{3}{2}\delta < r \quad \text{implies } X < 0. \tag{27}$$

2) Show that conditon (26) and (27) hold We now show that (26) and (27) are sufficient conditions to obtain an unambigious sign for X*:*

a) From (26) we obtain $\lambda < 3/2$ and from (27) we obtain $\lambda > 3/2$ the latter case will be in brackets: [<]

$$\begin{aligned}\frac{3}{4}\sigma^2 + 3\delta &> [<]\, 2r \\ \frac{3}{4} + 3\frac{\delta}{\sigma^2} + \frac{\delta^2}{\sigma^4} &> [<]\, \frac{\delta^2}{\sigma^4} + \frac{2r}{\sigma^2} \\ \frac{3}{4} + 2\frac{\delta}{\sigma^2} + \frac{\delta^2}{\sigma^4} &> [<] - \frac{\delta}{\sigma^2} + \frac{\delta^2}{\sigma^4} + \frac{2r}{\sigma^2}\end{aligned}$$

$$
\begin{aligned}
1+2\frac{\delta}{\sigma^2}+\left(\frac{\delta}{\sigma^2}\right)^2 &> [<]\,\frac{1}{4}-\frac{\delta}{\sigma^2}+\frac{\delta^2}{\sigma^4}+\frac{2r}{\sigma^2}\\
\left(1+\frac{\delta}{\sigma^2}\right)\left(1+\frac{\delta}{\sigma^2}\right) &> [<]\left(\frac{1}{2}-\frac{\delta}{\sigma^2}\right)^2+\frac{2r}{\sigma^2}\\
\left(1+\frac{\delta}{\sigma^2}\right)^2 &> [<]\left[\left(\frac{1}{2}-\frac{\delta}{\sigma^2}\right)^2+\frac{2r}{\sigma^2}\right]
\end{aligned}
$$

$$
\begin{aligned}
1+\frac{\delta}{\sigma^2} &> [<]\left[\left(\frac{1}{2}-\frac{\delta}{\sigma^2}\right)^2+\frac{2r}{\sigma^2}\right]^{\frac{1}{2}}\\
\frac{3}{2} &> [<]\,\frac{1}{2}-\frac{\delta}{\sigma^2}+\left[\left(\frac{1}{2}-\frac{\delta}{\sigma^2}\right)^2+\frac{2r}{\sigma^2}\right]^{\frac{1}{2}}
\end{aligned}
$$

As $\lambda=\frac{1}{2}-\frac{\delta}{\sigma^2}+\left[\left(\frac{1}{2}-\frac{\delta}{\sigma^2}\right)^2+\frac{2r}{\sigma^2}\right]^{\frac{1}{2}}$ we proved that the value of λ depends on the conditions (26) and (27). Therefore we get

$$\tfrac{3}{2}>\lambda \quad \text{for } \tfrac{3}{4}\sigma^2+3\delta>2r \tag{28}$$

$$\tfrac{3}{2}<\lambda \quad \text{for } \tfrac{3}{4}\sigma^2+3\delta<2r \tag{29}$$

b) Now we apply this condition for X:

$$X=\frac{\left[\left(\frac{1}{2}-\frac{\delta}{\sigma^2}\right)^2+\frac{2r}{\sigma^2}\right]^{-\frac{1}{2}}\frac{1}{\sigma^2}}{\lambda(\lambda-1)}-\frac{1}{r-\alpha}>[<]\,0$$

$$
\begin{aligned}
\frac{\left[\left(\frac{1}{2}-\frac{\delta}{\sigma^2}\right)^2+\frac{2r}{\sigma^2}\right]^{-\frac{1}{2}}\frac{1}{\sigma^2}}{\lambda(\lambda-1)} &> [<]\,\frac{1}{r-\alpha}\\
\frac{(r-\alpha)\left[\left(\frac{1}{2}-\frac{\delta}{\sigma^2}\right)^2+\frac{2r}{\sigma^2}\right]^{-\frac{1}{2}}\frac{1}{\sigma^2}}{\lambda(\lambda-1)} &> [<]\,1
\end{aligned}
$$

$$
\begin{aligned}
\frac{(r-\alpha)}{\lambda(\lambda-1)}\frac{1}{\sigma^2} &> [<]\left[\left(\frac{1}{2}-\frac{\delta}{\sigma^2}\right)^2+\frac{2r}{\sigma^2}\right]^{\frac{1}{2}}\\
\frac{(r-\alpha)}{\lambda(\lambda-1)\delta}\frac{\delta}{\sigma^2}-\frac{\delta}{\sigma^2}+\frac{1}{2} &> [<]\,\frac{1}{2}-\frac{\delta}{\sigma^2}+\left[\left(\frac{1}{2}-\frac{\delta}{\sigma^2}\right)^2+\frac{2r}{\sigma^2}\right]^{\frac{1}{2}}\\
\left[\frac{(r-\alpha)}{\lambda(\lambda-1)\delta}-1\right]\frac{\delta}{\sigma^2} &> [<]\,\lambda-\frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
\frac{(r-\alpha)}{\lambda(\lambda-1)\delta}-1 &> [<]\left(\lambda-\frac{1}{2}\right)\frac{\sigma^2}{\delta}\\
\frac{(r-\alpha)}{\delta}-\lambda(\lambda-1) &> [<]\left(\lambda-\frac{1}{2}\right)\frac{\sigma^2\lambda(\lambda-1)}{\delta}\\
(r-\alpha) &> [<]\,\lambda(\lambda-1)\left[\left(\lambda-\frac{1}{2}\right)\sigma^2+\delta\right]\\
r &> [<]\,\lambda(\lambda-1)\left[\left(\lambda-\frac{1}{2}\right)\sigma^2+\delta\right]+\alpha
\end{aligned}
$$

With the conditions (29) and (28) we know $\frac{3}{2}>\lambda$ for $\frac{3}{4}\sigma^2+3\delta>2r$ and $\frac{3}{2}<\lambda$ for $\frac{3}{4}\sigma^2+3\delta<2r$. Therefore we can check if we find true conditions for $X>0, X<0$ using the highest/lowest value of λ .

$$
\begin{aligned}
\frac{(r-\alpha)}{\delta} &> (<)\,\frac{3}{2}(\frac{3}{2}-1)\left[\left(\frac{3}{2}-\frac{1}{2}\right)\frac{\sigma^2}{\delta}+1\right]\\
& \quad (<)\,\frac{3}{2}(\frac{1}{2})\left[\left(\frac{2}{2}\right)\frac{\sigma^2}{\delta}+1\right]\\
&> (<)\,\frac{3}{4}(1)\left[\frac{2}{2}\frac{\sigma^2}{\delta}+1\right]\\
(r-\alpha) &> (<)\,\frac{3}{4}\left[\sigma^2+\delta\right]
\end{aligned}
$$

(i) conditions for $X>0$:

$$
\begin{aligned}
r &> \frac{3}{4}\left[\sigma^2+\delta\right]+\alpha\\
\frac{3}{8}\sigma^2+\frac{3}{2}\delta &> r \quad \text{see} \quad \text{above}\\
\frac{3}{8}\sigma^2+\frac{3}{2}\delta &> r>\frac{3}{4}\left[\sigma^2+\delta\right]+\alpha
\end{aligned}
$$

$$
\begin{aligned}
\Delta r &= \frac{3}{8}\sigma^2+\frac{3}{2}\delta-\frac{3}{4}\sigma^2-\frac{3}{4}\delta-\alpha>0\\
\Delta r &= -\frac{3}{8}\sigma^2+\frac{3}{4}\delta-\alpha>0
\end{aligned}
$$

$$
\begin{aligned}
3\delta &> \frac{3}{2}\sigma^2+4\alpha\\
\delta &> \frac{1}{2}\sigma^2+\frac{4}{3}\alpha
\end{aligned}
$$

(i) conditions for $X<0$:

$$
\begin{aligned}
r &< \frac{3}{4}\left[\sigma^2 + \delta\right] + \alpha \\
\frac{3}{8}\sigma^2 + \frac{3}{2}\delta &< r \quad \text{see} \quad (26) \\
\frac{3}{4}\left[\sigma^2 + \delta\right] + \alpha &> r > \frac{3}{8}\sigma^2 + \frac{3}{2}\delta
\end{aligned}
$$

$$
\begin{aligned}
\Delta r &= \frac{3}{4}\left[\sigma^2 + \delta\right] + \alpha - \frac{3}{8}\sigma^2 - \frac{3}{2}\delta > 0 \\
&= \frac{6}{8}\sigma^2 + \frac{3}{4}\delta + \alpha - \frac{3}{8}\sigma^2 - \frac{6}{4}\delta > 0 \\
\Delta r &= \frac{3}{8}\sigma^2 - \frac{3}{4}\delta + \alpha > 0
\end{aligned}
$$

$$
\delta < \frac{1}{2}\sigma^2 + \frac{4}{3}\alpha
$$

As we can see there is a feasible combination of δ, σ^2, and α satisfying this condition. The assumption that $X > 0$ and $X < 0$ is proven under the derived conditions (26) and (27).