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On Existence and Properties of Pure-strategy Equilibria under Contests

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Abstract

The use of 'ratio form' probability of success function dominates the existing literature on contests. Very few works have focused on the 'difference form' functions, notwithstanding their robust theoretical foundations and intuitive appeal in several contexts. Assuming the cost of efforts to be linear, Hirshleifer (1989) and Baik (1998) have argued that under the difference form contests, there is no interior pure strategy Nash equilibrium. In contrast, existence of interior pure strategy Nash equilibrium is well known for the ratio form contest functions. In this paper we use strictly convex cost functions and demonstrate existence of pure strategy Nash equilibrium for the difference form. Moreover, we show that several properties of equilibria and the comparative statics for the difference form closely resemble those for the ratio form. However, unlike the ratio form, under a difference form contest the existence of pure strategy Nash equilibrium is sensitive to the value of the prize.

1 Introduction

In several contexts, economic agents compete with one another for a 'prize'. For example, employees compete for promotion and higher wage. Sport persons compete for awards. Firms and advertising agencies compete for the market share. The research and development firms compete for patents. The litigants compete for a favorable court award. Politicians contest elections for various offices etc.

Given their relevance to several contexts, the literature on contests is extensive. Various types of contests have been modeled and thoroughly examined in the literature. For instances, Hirshleifer (1989) models battles/wars; Snyder (1989), Hillman & Riley (1989), Coughlin (1990), Baron (1994), Skaperdas & Grofman (1995) study political and electoral competition; litigation contests are modeled in Robson & Skaperdas

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(2008), Wärneryd (2000); patent races and innovation tournaments are studied in Loury (1979), Reinganum (1989), Nti (1997), Fullerton & McAfee (1999), Baye & Hoppe (2003); Szymanski (2003) examines sports, etc.

Formally, a contest is characterized by the choice of the probability of success function along with the cost of effort functions for the players. The probability of success for a player depends on levels of efforts put in by different contestants. Depending on the context, effort could mean the action taken or the time spent at talk. It could also mean the amount of money spent or investment made by an agent. A contest can be asymmetric in terms of the relative 'strength' of the players - a player might have natural advantages over the others. In addition, the probability of success may depend on the nature of underlying uncertainty, which can vary from context to context.

The most commonly used probability of success function is known as the *Tullock* or the *ratio form* function. As the name indicates, under the ratio form, probability of win for a player is the ratio of effort provided by her, divided by the total supply of efforts across all players.[1] The ratio form contest probability functions (CPFs) are used to study various types of rent-seeking contests, innovation tournaments, and patent-races. Competition games in these situations have been shown to be strategically equivalent to the ratio form contests. (SeeBaye & Hoppe (2003)).

The other form of CPF used in the literature is called the *difference form*. Under a difference form CPF, winning probabilities are functions of the difference in the level of efforts.(See Hirshleifer (1989), Baik (1998),Che & Gale (2000) etc.) This probability of contest function has been shown to be natural choice for model settings like intra-household bargaining over allocation of time and other resources, wars/battles, etc. (See Gersbach & Haller (2009), Hirshleifer (1989)). A difference form is also of interest when there is lack of commitment on the part of the contest designer to a specific CPF, once contenders have already exerted their efforts[2].

The ratio form as well as the difference form CPFs have strong theoretical foundations. Both classes have different but equally appealing axiomatic foundations[3]. Nonetheless, the literature on contests is dominated by the use of the ratio form CPFs. For the ratio forms, the literature has extensively examined properties of the equilibria under symmetric as well as asymmetric information structures. Several works have demonstrated the existence of an interior pure strategy Nash equilibrium for the ratio form under reasonable assumptions and different information structures.[4] Existence

[1]Skaperdas (1996), Baye et al. (1994), Cornes & Hartley (2005), Snyder (1989), Szymanski & Valletti (2005), Schweinzer & Segev (2012) etc.

[2]Corchón & Dahm (2010)

[3]See the next section

[4]Baye et al. (1994), Szidarovszky & Okuguchi (1997), Cornes & Hartley (2005), Wärneryd (2003)

of a pure strategy equilibria under the ratio form CPFs has enabled researchers to study their properties and comparative statics. Indeed, a large number of works have analyzed the comparative statics for the ratio form, to examine effects of starting advantage, value of prize and information structure on equilibrium effort levels and pay-offs for the agents.

In contrast, very few works exist on the difference form functions. Assuming the cost of efforts to be linear, Hirshleifer (1989) and Baik (1998) have shown that under the difference form CPFs there does not exist an interior pure strategy Nash equilibrium[5]. It seems that the existing claims about the nature of equilibrium under the difference form CPFs is a reason why these functions are under studied, even though they have equally robust theoretical foundations and plausible intuitive appeal.

The question is whether the non-existence of an interior pure strategy Nash equilibrium under difference form functions is due to some intrinsic feature of these CPFs; or, it is on account of the assumption that costs are linear. We relax the assumption of linear cost and show that when costs are strictly convex, 'impossibility theorem' in Hirshleifer (1989) and Baik (1998) breaks down. We demonstrate existence of interior pure strategy Nash equilibrium for various types of difference form CPFs.

In this paper, we assume strictly convex costs of efforts. The assumption of strictly convex costs is not only an analytical possibility, but is mainstream. A large number of economic models, including contests and tournaments make this assumption e.g. Holmstrom (1982), Grund & Sliwka (2005), Moldovanu & Sela (2006), Demougin & Fluet (2003), Harbring & Irlenbusch (2003), Cornes & Hartley (2012), Esteban & Sákovics (2003), Einy et al. (2013), Ewerhart & Quartieri (2015) etc. St-Pierre (2016) show the importance of convexity in the cost function on aggregate performance in a tournament model. Moldovanu & Sela (2001) consider convex cost functions most important from application point of view as it can make several positive prizes optimal unlike linear or concave costs in a contest with multiple non identical prizes. Indeed, in several contexts, the assumption of strictly convex costs is more plausible than the linear costs. For instance, when efforts are non-monetary, say in terms of time spent as described in Cubel (2014).

Further, we show that the properties of the equilibria and the comparative statics for the difference form CPFs closely resemble those for the ratio form. However, there are a few interesting differences as well. We show that when contests are asymmetric with respect to innate ability or productivity, then under the difference form, the existence of pure strategy Nash equilibrium is sensitive to the size of the prize.

The rest of the article is organized as follows. In section 2, we review the existing

etc.

[5]Also see Che & Gale (2000), Hwang (2009)

literature. In section 3 we describe the structure of the model and introduce several classes of contest probability functions(CPFs) we study. In section 4 we examine existence of equilibria for various classes of CPFs and also their properties. Section 5 analysis the comparative static results for difference form CPFs and compares them to the ratio form. Finally in section 6, we present conclusions.

2 Contest Functions: Literature Review

A contest is characterized by the choice of the contest probability function (CPF) along with the cost of effort functions for the players. A CPF, commonly known as contest success function, maps agents' efforts into their probabilities of win.
The most commonly used CPF is the *logit* form. Some of the many articles that work with logit form include Dixit (1987), Snyder (1989), Hurley & Shogren (1998), Szidarovszky & Okuguchi (1997), Dahm et al. (2005) etc. Logit form itself can take several forms. The one most extensively used is the Tullock or the *ratio form* function. For two-player contests, a generic version of the ratio form function defines probability of win for the first player, $p_1(.)$ as:

$$p_1(.) = \frac{\theta_1 e_1^m}{\theta_1 e_1^m + \theta_2 e_2^m}$$

where $m > 0$, e_1, e_2 are the efforts put in by the players, θ_1 and θ_2 are parameters that determine relative advantages of the players. The probability of win for the second player is $p_2(.) = 1 - p_1(.)$. The existing literature has used special cases of the ratio form as defined above. Apart from few exceptions, it is assumed that $m = 1$ e.g. Snyder (1989), Nitzan (1991), Paul & Wilhite (1991), Caruso (2006), Caruso (2007), Fu & Lu (2012), Wärneryd (2003), Fey (2008). The ratio form CPFs are used to study various types of the rent-seeking and lobbying contests (Konrad (2000)), sports tournaments (Szymanski (2003), Hoehn & Szymanski (2010)), electoral competitions (Snyder (1989)), political conflicts and power struggles (Hirshleifer (1991), Hirshleifer (1995), Anderton (2000)), innovation tournaments and patent-race (Baye & Hoppe (2003)). Competition games in these situations have been shown to be strategically equivalent to the ratio form contests by Baye & Hoppe (2003). This CPF is also used in various experimental group contest games (Abbink et al. (2010), Ahn et al. (2011), Abbink et al. (2012)).

The other form of CPF used in the literature is called the *difference form*. Under a difference form CPF, winning probabilities are functions of the difference in the level of efforts. Under a typical difference form, $p_1(.)$ is given by:

$$p_1(.) = \frac{exp(me_1)}{exp(me_1) + exp(me_2)} \equiv \frac{1}{1 + exp(me_2 - me_1)}, \tag{1}$$

where $m > 0$ and $p_2(.) = 1 - p_1(.)$. See e.g. the CPFs used in Hirshleifer (1989), Baik (1998), Che & Gale (2000), Polishchuk & Tonis (2013), Anderton (2000), Gersbach &

Haller (2009), etc. This type of CSF also culminates from using a mechanism design approach when contestants have private information about value of prize (Polishchuk & Tonis (2013)).[6]

There exists yet another form of the difference form CPFs, which defines

$$p_1(.) = f(d) \tag{2}$$

where $d = me_1 - e_2$, $m > 0$, and $f(d)$ is continuous and twice differentiable such that $f'(d) > 0$. This CPF is described by Baik (1998). Besides, there are some other forms of contest success functions. However, these functions have properties starkly different from the CPFs we want to analyze in this article. For instance, see Skaperdas et al. (2013); Beviá & Corchón (2015).

The literature on this subject shows that the above described CPFs have micro-founded. Corchón & Dahm (2010) for example, suggest that the general two player logit form can be rationalized if the decision maker has multiplicative pair of utility functions. He uses these utility functions to decide the winner given state of nature known only to him. In Dahm et al. (2005), logit form emerges as an outcome of a model setting where players are uncertain about the type of the contest administrator.

The existing literature on the subject also provides stochastic foundations underlying the above mentioned CPFs. For instance, the ratio form CPFs emerges when the error term, ϵ, follows inverse exponential distributions as seen in Jia (2008). Jia et al. (2013) argues that the approach in Jia (2008) can be extended to provide stochastic foundations for symmetric and asymmetric versions of general logit form as well. Similarly, McFadden et al. (1973) show that difference form functions are derived when ϵ follows extreme value distribution.

Besides, these functions also have axiomatic foundations as shown in Skaperdas (1996) for logit and ratio forms, Clark & Riis (1998) for asymmetric ratio forms etc. Axiomatic route to the foundations of logit, ratio and difference forms are also provided in Ewerhart (2015). The ratio and difference form functions have different but equally appealing axiomatic foundations as seen in Jia et al. (2013). Skaperdas (1996) and Jia et al. (2013) argue for instance that the ratio form can be derived from class of functions satisfying homogeneity property, i.e., $p_i(ce_i, ce_{-i}) = p_i(e_i, e_{-i}) \; \forall \; c > 0$. The difference form functions exhibit an equally interesting property: for each player the probability of winning does not change if the level of effort of each player increases by a constant amount. Formally, for CPF defined in (1), we have $p_i(e_i + c, e_{-i} + c) = p_i(e_i, e_{-i})$, $\forall \; c$ such that $e_j + c > 0$.

[6] Even literature from empirical IO makes use of the difference form function as demand function emerging from stochastic utility functions with error term following extreme value distribution. However, the problem considered in these articles is different from what we deal with here.

Nonetheless, the literature contests are dominated by the use of the ratio form CPFs. For the ratio forms, the literature has extensively examined properties of the equilibria under symmetric as well as asymmetric information structures e.g. Baye et al. (1994), Szidarovszky & Okuguchi (1997), Cornes & Hartley (2005), Chowdhury & Sheremeta (2011), Wärneryd (2003). Besides, a large number of works like Nti (1999), Baik (2004), Snyder (1989), Fey (2008) have analyzed the comparative statics for the ratio form, to examine effects of starting advantage, value of prize and information structure on equilibrium effort levels and pay-offs for the agents. In contrast, very few works exist on the difference form functions. See Hirshleifer (1989), Baik (1998), Che & Gale (2000), Hwang (2009), and Baye et al. (1994). To our knowledge, there is no study on the comparative statistics for the class of difference form function we consider in the article including (1). In this paper, we fill this gap in the literature.

The question is whether the non-existence of an interior pure strategy Nash equilibrium under difference form functions is due to some intrinsic feature of these CPFs; or, it is on account of the assumption that costs are linear. We show that when costs are strictly convex, 'impossibility theorem' in Hirshleifer (1989) and Baik (1998) break down. We demonstrate existence of an interior pure strategy Nash equilibrium for two different classes of difference form CPFs, including the functions defined in (1) and (2).

3 Model

Two players compete for a prize, which is a random variable denoted by V. The random variable V is drawn from support $[\underline{v}, \bar{v}]$. Let any particular realization of V be denoted by v and let $F(v)$ and $f(v)$, respectively, be the distribution and the density functions of V.

A contest can be modeled as a comparison of the 'outputs' produced by the competing agents; agent with the highest output wins the prize. In general, the output depends on the effort by the agent and a noise term. Let, $Q_i = q_i(e_i, \epsilon_i)$ be the output production function for agent i, where e_i is the effort put in by player and ϵ_i is the relevant error term for $i = 1, 2$.[7] The Probability of win for the agent i, $p_i(.)$, is given by $p_i(e_i, e_j) = \Pr[Q_i > Q_j] = \Pr[q_i(e_i, \epsilon_i) > q_j(e_j, \epsilon_j)]$, where $i, j = 1, 2$ and $i \neq j$. Clearly, $p_i(.)$ is a function of the efforts provided by the two players along with the parameters of the distribution of the two error terms. In particular, depending on $q_i(.)$ and distribution of ϵ_i, $p_i()$ can take several forms like the ratio form and the difference form as described in the previous section[8].

For simplicity, let $p_1(.) = p(.)$ and therefore $p_2(.) = 1 - p(.)$. The general form

[7] The term 'outputs' can stand for the actual output, the quantity of sale, evidence, etc.

[8] See Konrad et al. (2009), and Jia et al. (2013)

logit probability of success function is given by: $p(.) = \theta_1\phi_1(e_1)/[\theta_1\phi_1(e_1) + \theta_2\phi_2(e_2)]$, where $\theta_i > 0$, $\phi_i(e_i) \geq 0$, $\phi_i^{'}(e_i) > 0$, $i = 1, 2$ and $\phi_1(e_1) > 0$ or $\phi_2(e_2) > 0$. If $\phi_1(e_1) = \phi_2(e_2) = 0$ then $p(.) \in (0, 1)$. This CPF can be re-written as: $p(.) = \theta\phi_1(e_1)/[\theta\phi_1(e_1) + \phi_2(e_2)]$, where $\theta = \theta_1/\theta_2$. Clearly, $\theta > 0$. Note that when $\phi_1(e_1) = \phi_2(e_2)$, i.e., when the two players provide same amount of output, $p(.) = p_1(.) \lesseqgtr 1/2$ iff $\theta_1 \lesseqgtr \theta_2$, i.e., $\theta \lesseqgtr 1$. Thus $\theta \neq 1$ is a source of asymmetry in the contest. Specifically, θ is a measure of 'natural advantage' of player 1 over 2.

One can think of $\phi_i(e_i)$ as output/performance/evidence provided by party i. It can also be interpreted as the impact function of player i. A completely different interpretation for $\phi_i(.)$ comes from the literature on micro-foundations which interprets it as a component of the contest administrators utility function based on which he decides the winner. In such setups $\phi_i(.)$ can also be interpreted as bribes/transfers to the administrators[9].

A more general version of the logit form can be re-written as:

$$p(.) = \frac{\theta\phi_1(e_1, m)}{\theta\phi_1(e_1, m) + \phi_2(e_2, n)}, \tag{3}$$

where $m, n \in \Re_{++}$. Let this class of contest probability functions be denoted by $\mathbb{P}_L$. Formally,

$$\mathbb{P}_L = \{p(.)|p(.) \text{ is defined in (3)}\}.^{10} \tag{4}$$

The class $\mathbb{P}_L$ captures the idea that the 'output' function $\phi_i()$ may itself be a function of some innate ability (productivity) of the player i. If $\phi_1() > \phi_2()$ for given e, then player 1 is more productive than player 2 because for any given level of effort, the output produced is higher for player 1. For example, we can have $\phi_1(e, m) = e^m$ and $\phi_2(e, n) = e^n$; $m > n$ (for $e > 1$) means player 1 is more productive/able as compared to player 2 and opposite will hold for $n > m$. We can also have $\phi_1(e, m) = \exp(me)$, $\phi_2(e, n) = \exp(ne)$ where $m \lesseqgtr n$ indicates relative productivity.

For $p(.) \in \mathbb{P}_L$, the asymmetry of the contest, i.e., the relative advantage of the players depend on θ as well on m versus n.

The existing literature has mostly focused on the asymmetry as represented by θ here, and not the one on account of m and n. We will model and examine both sources of asymmetry. We show that the two types of asymmetry have very different impact on equilibrium efforts, costs and probabilities of success.

Some subclasses of $\mathbb{P}_L$ which are of special interest.

[9]Dahm et al. (2005)

[10]This class of logit forms allows different functional forms for $\phi_1(.)$ and $\phi_2(.)$ e.g. $\phi_1(e_1) = e_1^m$ and $\phi_2(e_2) = \exp(ne_2)$, etc.

$$\mathbb{P}_R = \left\{ p(.)|p(.) = \frac{\theta e_1^m}{\theta e_1^m + e_2^n} \mid \theta \in \Re_{++};\ m, n \in (0, 2] \right\} \quad (5)$$

$$\mathbb{P}_E = \left\{ p(.)|p(.) = \frac{\theta \exp(me_1)}{\theta \exp(me_1) + \exp(ne_2)} \mid \theta, m, n \in \Re_{++} \right\} \quad (6)$$

$\mathbb{P}_R$ is the class of 'ratio forms'[11]. $\mathbb{P}_E$ is the class of 'exponential difference forms'[12]. Both $\mathbb{P}_R$ and $\mathbb{P}_E$ are more general than the corresponding forms in the literature[13]. Also, $\mathbb{P}_E \cup \mathbb{P}_R \subset \mathbb{P}_L$[14].

Besides, there is another class of the 'difference form' functions introduced by Baik (1998). We denote this class by

$$\mathbb{P}_D = \left\{ p(.)|p(.) = f(d),\ \text{where } d = me_1 - e_2;\ m \in \Re_{++} \right\} \quad (7)$$

Following Baik (1998), we assume that $f(d)$ is twice continuously differentiable and is such that $f'(d) > 0\ \forall\ d \in \Re$; $f''(d) \gtreqless 0\ \forall\ d \lesseqgtr 0$; $f(0) = 1/2$; $0 < f(d) < 1$ and $f(-d) = 1 - f(d)$.

Clearly, $\mathbb{P}_E \cap \mathbb{P}_R = \phi$. However $\mathbb{P}_E \cap \mathbb{P}_D \neq \phi$. For instance, $\frac{\exp(me_1)}{\exp(me_1)+\exp(e_2)} \in \mathbb{P}_E \cap \mathbb{P}_D$. If $p \in \mathbb{P}_E$ and $n = 1$, then $p \in \mathbb{P}_D$ i.e. $n = 1 \Rightarrow \mathbb{P}_E \subseteq \mathbb{P}_D$. However, we show that the assumption $n = 1$, comes at the cost of generality. In general, neither $\mathbb{P}_E \subseteq \mathbb{P}_D$ nor $\mathbb{P}_D \subseteq \mathbb{P}_E$. That is, $\mathbb{P}_E$ and $\mathbb{P}_D$ denote different class of difference forms. As illustrations, note that $\frac{\theta \exp(me_1)}{\theta \exp(me_1)+\exp(ne_2)} \in \mathbb{P}_E$ but $\notin \mathbb{P}_D$. In contrast, $p = f(d) = \frac{\lambda}{1+c^{-kd}} + \frac{1-\lambda}{2}$ belongs to $\mathbb{P}_D$ but not to $\mathbb{P}_E$. Similarly, when $p = f(d) = (\sqrt{2\pi}q)^{-1} \int_{-\infty}^{d} \exp(\frac{-z^2}{2q^2})dz$ where $c > 1,\ q > 0,\ k > 0,\ 0 < \lambda < 1$ then $p \in \mathbb{P}_D$ but $\notin \mathbb{P}_E$.

To sum up, the probability of success function can be written $p(\theta, m, n, e_1, e_2)$ and satisfies the following properties: $\frac{\partial p(.)}{\partial e_1} > 0,\ \frac{\partial p(.)}{\partial e_2} < 0,\ \frac{\partial p(.)}{\partial \theta} > 0$.

If the value of the prize is known to both the parties with certainty (perfect information), say when efforts are chosen after the random variable has been realized and observed by the both players, then for any given effort level e_2 opted by 2, individual 1 chooses e_1 to solve

$$\max_{e_1} \{p(\theta, m, n, e_1, e_2)v - \psi_1(e_1)\} \quad (8)$$

[11]The literature has generally considered the cases with $m = n = 1$. In general $m = n$ case can be derived when $q_i(e_i, \epsilon_i) = e_i.\epsilon_i$ and ϵ_i follows inverse exponential distribution

[12]the $m = n$ case can be derived if $q_i(e_i, \epsilon_i) = e_i + \epsilon_i$ and ϵ_i follows extreme value distribution (for details see Jia et al. (2013))

[13]See Section 2.

[14]e.g. $\theta \log(me_1 + 1)/[\theta log(me_1 + 1) + \log(ne_2 + 1)] \in \mathbb{P}_L$ but $\notin \mathbb{P}_R$ and $\notin \mathbb{P}_E$

where $\psi_1(e_1)$ is continuously differentiable, $\psi_1^{'}(e_1) > 0$ $e_1 > 0$, $\psi_1'(0) = 0$, $\lim_{e_1 \to \infty} \psi_1'(e_1) \to \infty$[15] and $\psi_1^{''}(e_1) > 0$ $e_1 > 0$. On the other hand, if the value of the random variable V is not known to either player (symmetric uncertainty), individual 1 chooses e_1 to solve

$$\max_{e_1} \{p(\theta, m, n, e_1, e_2)E(V) - \psi_1(e_1)\}, \tag{9}$$

where $E(V) = \int_{\underline{v}}^{\bar{v}} v dF(v)$. The optimization problem for the second player are defined similarly.

In the next section, we work with the above described classes of CPFs. The aim is to characterize the conditions for existence of interior pure strategy Nash equilibrium for these classes.

4 Equilibria: Existence and Properties

Now we examine the existence and properties of interior pure strategy Nash equilibrium for various classes of CPFs in the presence of perfect information about value of prize, v.

4.1 Logit Form CPF

In this section we consider $p \in \mathbb{P}_L$. For any given effort level e_2, player 1 chooses e_1 to solve

$$\max_{e_1} \left\{ \frac{\theta\phi_1(e_1)}{\theta\phi_1(e_1) + \phi_2(e_2)} v - \psi_1(e_1) \right\} \tag{10}$$

where $\phi_1(e_1) \geq 0$ and $\phi_1'(e_1) > 0$. The corresponding problem for player 2 is analogous. The two FOCs can then be written as:

$$\frac{\phi_i^{'}(e_i)}{\phi_i(e_i)} p_i(1 - p_i)v - \psi_i^{'}(e_i) = 0, \ i = 1, 2 \tag{11}$$

In view of the properties of $\phi_i(e_i)$, for any given $e_j > 0$, $e_i = 0$ can not solve FOC for player i. Hence one-sided corner solutions are ruled out[16].

When $p \in \mathbb{P}_R$, let $x_i = \psi_i(e_i)$. Since $\psi_i(.)$ is a monotonic and convex function we can write $y_i(x_i) = \psi_i^{-1}(x_i)$ where $y_i(x_i) = y_i(\psi_i(e_i)) = e_i$. Thus we can rewrite player 1's optimization problem as

$$\max_{x_1} \left\{ \frac{\theta(y_1(x_1))^m}{\theta(y_1(x_1))^m + (y_2(x_2))^n} v - x_1 \right\} \tag{12}$$

[15] Similar assumptions are made for $\psi_2(e_2)$

[16] For $p \in \mathbb{P}_R$ or $p \in \mathbb{P}_E$ even $(0, 0)$ can be ruled out

Clearly $y_i'(x_i) > 0$ and $y_i''(x_i) < 0$.
Using expression (11), for a general $p \in \mathbb{P}_L$, the FOC for player i can be defined as $g_i(.) = 0$, $i = 1, 2$ i.e., given m, n, θ, v and e_j $j \neq i$ opted by the other player, where

$$\begin{aligned} g_1(m,n,\theta,v,e_1,e_2) &= \frac{m\theta \exp{(me_1 + ne_2)}}{[\theta \exp{(me_1)} + \exp{(ne_2)}]^2} v - \psi_1^{'}(e_1) && (13) \\ &= mp(1-p)v - \psi_1^{'}(e_1). && (14) \end{aligned}$$

$$\begin{aligned} g_2(m,n,\theta,v,e_1,e_2) &= \frac{n\theta \exp{(me_1 + ne_2)}}{[\theta \exp{(me_1)} + \exp{(ne_2)}]^2} v - \psi_2^{'}(e_2) && (15) \\ &= np(1-p)v - \psi_2^{'}(e_2). && (16) \end{aligned}$$

e_1^* and e_2^*, as implied by (14) and (16), would be such that:

$$\frac{m}{n} = \frac{\psi_1^{'}(e_1^*)}{\psi_2^{'}(e_2^*)} \tag{17}$$

From (13), it is easy to check that $e_1 = 0$ cannot solve FOC1 for any $e_2 \geq 0$. Additionally, from (15), it can be seen that $e_2 = 0$ cannot solve FOC2 for any $e_1 \geq 0$. Thus we can rule out corner solutions where either players' effort is zero and hence we restrict attention to $e_1, e_2 > 0$.

For given m, n, θ and v, $g_1(.)$ is a continuously differentiable function. If we can ensure that $\forall e_1, e_2 > 0$ $||\frac{\partial g_1}{\partial e_1}||$ is bounded away from zero, we can find a continuous (smooth) function $e_1^*(e_2)$ such that $g_1(m,n,\theta,v,e_1^*(e_2),e_2) = 0$.[17] Here $\frac{\partial g_1}{\partial e_1} = m^2 vp(.)(1-p(.))(1-2p(.)) - \psi_1^{''}(e_1)$ where $p(.)(1-p(.))(1-2p(.))$ is bounded above. Moreover $\psi_1^{''}(e_1) > 0$ $\forall e_1 > 0$ and therefore for sufficiently small values of $m^2 v$, $\frac{\partial g_1}{\partial e_1} < 0$ $\forall e_1, e_2 > 0$. As an illustration consider the case when $\psi_1(e_1) = e_1^2/2$. Then $m^2 v < 6\sqrt{3}$ implies $\frac{\partial g_1}{\partial e_1} < 0$ $\forall e_1, e_2 > 0$. Thus we have a continuous (smooth) function $e_1^*(e_2)$ such that $g_1(m,n,\theta,v,e_1^*(e_2),e_2) = 0$. Similarly, since $\frac{\partial g_2}{\partial e_2} = n^2 vp(.)(1-p(.))(2p(.)-1) - \psi_2^{''}(e_2)$, for sufficiently small values of $n^2 v$, we can derive a continuous function $e_2^*(e_1)$ such that $g_2(m,n,\theta,v,e_1,e_2^*(e_1)) = 0$.

$e_2^*(e_1)$ can be interpreted as the effort of player 2 which solves FOC2 for given e_1. If there exists an $e_1 = \hat{e}_1 > 0$ such that $g_1(m,n,\theta,v,\hat{e}_1,e_2^*(\hat{e}_1)) = 0$, then $\hat{e}_1$ and $e_2^*(\hat{e}_1)$ will solve both the FOCs simultaneously. Given $m^2 v > 0$, $\lim_{e_1 \to 0^+} g_1(m,n,\theta,v,e_1,e_2^*(e_1)) > 0$, as $\lim_{e_1 \to 0} \psi_1'(e_1) \to 0$ and $\lim_{e_1 \to \infty} g_1(m,n,\theta,v,e_1,e_2^*(e_1)) \to -\infty$. Since $g_1(.)$ and $e_2^*(e_1)$ are continuous functions of e_1, at some $e_1 > 0$, $g_1(m,n,\theta,v,e_1,e_2^*(e_1)) = 0$. Thus we can conclude that under suitable parametric restrictions, there exists a pair of (e_1, e_2), say (e_1^*, e_2^*) which solves both the FOCs and is such that $e_1^*, e_2^* > 0$. In other words, for at least some $p \in \mathbb{P}_{\mathbb{L}}$, an interior solution (e_1^*, e_2^*) for the FOCs exist.

[17] Using Lemma 2 from Zhang & Ge (2006).

Furthermore, if the second order conditions (SOC's) are satisfied then $e_2^*(e_1)$ is a Nash equilibria of the contest.

Let $p(e_1^*, e_2^*) = p^*$. From (11) it follows that an interior pure strategy Nash equilibrium (e_1^*, e_2^*) will satisfy:

$$\frac{\phi_1(e_1^*)\psi_1'(e_1^*)}{\phi_1'(e_1^*)} = \frac{\phi_2(e_2^*)\psi_2'(e_2^*)}{\phi_2'(e_2^*)} = vp^*(1-p^*) \tag{18}$$

We can re-write the above as

$$\Phi_1(e_1^*)\psi_1'(e_1^*) = vp^*(1-p^*) = \Phi_2(e_2^*)\psi_2'(e_2^*), \tag{19}$$

where $\Phi_i(.) = \phi_i(.)/\phi_i'(.)$. Let,

$$\Psi_i(.) = \Phi_i(.)\psi_i'.$$

When $\phi_i(.)$ and $\psi_i(.)$, $i = 1, 2$, are such that $\Psi_1(.) = \Psi_2(.) = \Psi(.)$, at any interior Nash equilibrium, the following will hold:

$$\Psi(e_1^*) = \Psi(e_2^*). \tag{20}$$

Now, if $\Psi'(.) > 0$, then $e_1^* = e_2^*$ will also hold[18]. That is, we have the following result.

Proposition 1 *When $\Psi_1(.) = \delta\Psi_2(.)$, and $\Psi_i'(.) > 0$, $i = 1, 2$, an interior pure strategy Nash equilibrium will be symmetric iff $\delta = 1$.*

Note that $\Psi_1(.) = \Psi_2(.)$ holds for several different combinations of functional forms of $\phi_i(.)$, $i = 1, 2$ and $\psi_i(.)$, $i = 1, 2$. If $\phi_1(e) = \phi_2(e)$ and $\psi_1(e) = \psi_2(e)$ then $\Psi_1(.) = \Psi_2(.)$ trivially holds. For example $\phi_1(e) = \phi_2(e) = \exp e$ and $\psi_1(e) = \psi_2(e) = \frac{e^2}{2}$, $\Psi_1(e) = \Psi_2(e) = e$. Very different functional forms of $\phi_i(.)$, $\phi_j(.)$, $\psi_i(.)$ and $\psi_j(.)$ can also give us $\Psi_1(.) = \Psi_2(.)$. For example, when $\phi_i(e) = \exp e$ and $\psi_i(.) = \frac{e^2}{2}$, on one hand, and $\phi_j(e) = e^2$ and $\psi_j(e) = 2e$ give us $\Psi_i(e) = \Psi_j(e) = e$.

Furthermore, when $\phi_j(.) = \eta\phi_i(.)$, $\eta > 0$, $\Phi_i(.) = \Phi_j(.)$ holds, which in turn gives the following result.

Proposition 2 *Suppose, $\Psi_1(.) = \Psi_2(.) = \Psi(.)$, and $\Psi_i'(.) > 0$, $i = 1, 2$, and $\phi_2(.) = \eta\phi_1(.)$, $\eta > 0$ then an interior pure strategy Nash equilibrium is symmetric as well as unique.*

Proof: Assume there are multiple equilibria. Consider any two distinct N.E., say $(\bar{e}_1, \bar{e}_2)$ and $(\tilde{e}_1, \tilde{e}_2)$. In view of the above proposition, at any N.E., we have $\bar{e}_1 = \bar{e}_2$

[18]For convex cost functions, $\Psi'(.) > 0$ when $p \in \mathbb{P}_E$ or $p \in \mathbb{P}_R$.

and $\tilde{e}_1 = \tilde{e}_2$. Moreover, we have $\Psi(\bar{e}_1) = vp(\bar{e}_1, \bar{e}_2)(1 - p(\bar{e}_1, \bar{e}_2)) = \Psi(\bar{e}_2)$, and $\Psi(\tilde{e}_1) = vp(\tilde{e}_1, \tilde{e}_2)(1 - p(\tilde{e}_1, \tilde{e}_2)) = \Psi(\tilde{e}_2)$. However, when $\bar{e}_1 = \bar{e}_2$ and $\tilde{e}_1 = \tilde{e}_2$,

$$\phi_j(.) = \eta\phi_i(.) \Rightarrow [p(\bar{e}_1, \bar{e}_2) = p(\tilde{e}_1, \tilde{e}_2)].$$

Therefore, we must have $\Psi(\bar{e}_1) = \Psi(\tilde{e}_1)$ and $\Psi(\bar{e}_2) = \Psi(\tilde{e}_2)$. This give us: $\bar{e}_1 = \tilde{e}_1$ and $\bar{e}_2 = \tilde{e}_2$. That is, the equilibrium is unique.

Proposition 3 *Suppose* $\Psi_1(.) = \Psi_2(.) = \Psi(.)$ *and* $\Psi'(.) > 0$. *If* $p(e_1, e_2) = p(te_1, te_2)$ *for all* $t > 0$, *then an interior pure strategy Nash equilibrium will be symmetric and unique.*

Proof follows from the fact that when $p(.)$ is homogeneous function of degree 0, $\phi_2(.) = \eta\phi_1(.)$ will hold.

Propositions 1 to 3 give the properties that any pure strategy Nash equilibrium for a logit form CPF will satisfy, however we are yet to prove the existence of a Nash equilibrium. Szidarovszky & Okuguchi (1997) show that for a subclass of $\mathbb{P}_{\mathbb{L}}$, there exists a N.E. Specifically, they show that for $\phi_i(0) = 0$, $\phi_i'(e_i) > 0$, $\phi_i''(e_i) \leq 0$ and $\psi_i''(e_i) = 0, i = 1, 2$, a unique Nash equilibrium for the problem exists. For $p \in \mathbb{P}_{\mathbb{R}}$ with cost functions of the form $\psi_i(e_i) = e_i^k$ $k > 1$, (12) can be rewritten as

$$\max_{x_1} \left\{ \frac{\theta x_1^{\bar{m}}}{\theta x_1^{\bar{m}} + x_2^{\bar{n}}} v - x_1 \right\} \tag{21}$$

where $\bar{m} = m/k$ and $\bar{n} = n/k$. If $m, n \leq k$, results from Szidarovszky & Okuguchi (1997) can be used to show the existence of a unique interior pure strategy Nash equilibrium. Further, even for $k < m = n \leq 2k$ results from Baye et al. (1994) and Cornes & Hartley (2005) show that an interior pure strategy Nash equilibrium exists.

However, existence of an equilibrium when $p \in \mathbb{P}_{\mathbb{E}}$ cannot be inferred from the literature. As the following examples show depending on value of the prize and the parameters of contest, a Nash equilibrium may or may not exist. For these examples we take $\psi(e) = e^2/2$.

Example 1 Suppose $m = 1/2,\ n = 1,\ \theta = 1,\ v = 200$. The only pair of efforts that solves the FOC's is $(e_1^*, e_2^*) \approx (2.44, 4.88)$ which gives $p \approx 0.025$. However, in this case, we get $\frac{\partial g_1}{\partial e_1}\Big|_{e_1^*, e_2^*} = 0.16 > 0$ and $\frac{\partial g_2}{\partial e_2}\Big|_{e_1^*, e_2^*} = -5.64 < 0$. That is, the SOC does not hold for the first player. So, a Nash equilibrium does not exist.

Example 2 Let $m = 2,\ n = 1,\ \theta = 1,\ v = 100$. Now, the unique solution to FOCs is $(e_1^*, e_2^*) \approx (2.82, 1.41)$, which gives $p \approx 0.986$. Moreover, $\frac{\partial g_1}{\partial e_1}\Big|_{e_1^*, e_2^*} = -6.48 < 0$ $\frac{\partial g_2}{\partial e_2}\Big|_{e_1^*, e_2^*} = 0.37 > 0$ Again, a Nash equilibrium does not exist, as the SOC does not hold for the second player.

Example 3 Let $m = n = 1,\ \theta = 1/2,\ v = 100$. In this case, the pair that solves the FOC's is $(e_1^*, e_2^*) \approx (22.22, 22.22)$ which gives $p = 1/3$. However, $\frac{\partial g_1}{\partial e_1}\Big|_{e_1^*, e_2^*} = 6.407 > 0$, and $\frac{\partial g_2}{\partial e_2}\Big|_{e_1^*, e_2^*} = -8.407 < 0$. That is, there is no Nash equilibrium as the SOC does not hold for the first player.

Example 4 Suppose $m = 2,\ n = 1,\ \theta = 1,\ v = 25$. The FOCs yield $(e_1^*, e_2^*) \approx (2.06, 1.03)$ and $p \approx 0.95$. Moreover, $\frac{\partial g_1}{\partial e_1}\Big|_{e_1^*, e_2^*} = -4.77 < 0$, and $\frac{\partial g_2}{\partial e_2}\Big|_{e_1^*, e_2^*} = -0.06 < 0$. That is, the SOC hold for both players. Moreover, deviation to 0 effort is not profitable to the either player. So, $(2.06, 1.03)$ is a Nash equilibrium.

Example 5 Consider $m = 2\ n = 1,\ \theta = 2,\ v = 40$. In this case, it can be checked that $(1.97, 0.98)$ is a Nash equilibrium.

Below we show that under suitable parametric conditions a Nash equilibrium does exist. We have already discussed the conditions which are sufficient to ensure that a solution (e_1^*, e_2^*) to the FOCs exist. Next we check for SOCs in order to claim that (e_1^*, e_2^*) is the mutual best response. Let us assume $\psi_i(e) = \psi(e);\ i = 1, 2$. Now, the SOC's for the players are given by $\frac{\partial g_i}{\partial e_i}\Big|_{e_1^*, e_2^*} \leq 0\ \ i = 1, 2$ where, $g_1(.)$ and $g_2(.)$ are as defined above, and

$$\frac{\partial g_1}{\partial e_1} = m(1-2p)\frac{\partial p}{\partial e_1}v - \psi''(e_1) = m^2(1-2p)p(1-p)v - \psi''(e_1);$$

$$\frac{\partial g_2}{\partial e_2} = n(1-2p)\frac{\partial p}{\partial e_2}v - \psi''(e_2) = n^2(2p-1)p(1-p)v - \psi''(e_2).$$

First consider the case when $m = n$. In this case, since $\psi_1(.) = \psi_2(.)$ by assumption, from Proposition 2 we conclude that a solution to the FOCs is symmetric and unique. The solution is given by $e_1^* = e_2^* = \psi'^{-1}\big(\frac{mv\theta}{(\theta+1)^2}\big)$. Note that $e_1^* = e_2^* \implies p^* = \frac{\theta}{\theta+1}$. Thus the required SOCs can be re-written as:

$$\frac{\partial g_1}{\partial e_1}\Big|_{e_1^*, e_2^*} = m^2 v\frac{(1-\theta)\theta}{(1+\theta)^3} - \psi''(e_1^*(m, v, \theta)) \leq 0;$$

$$\frac{\partial g_2}{\partial e_2}\Big|_{e_1^*, e_2^*} = m^2 v\frac{(\theta-1)\theta}{(1+\theta)^3} - \psi''(e_2^*(m, v, \theta)) \leq 0$$

For $\theta = 1$, the SOCs for both players are trivially satisfied. If $\theta > 1$ then SOC1 is satisfied for all values of m and v but this is not necessarily the case with SOC2. When $\theta < 1$, SOC2 is satisfied for all values of m and v but the SOC1 may or may not hold. However, from the above SOCs it is clear that for any give set of parametric values, the SOCs hold for sufficiently small values of v.

Specifically, when $\psi(.) = e^2/2$, the sufficient conditions for the two SOCs to hold are: When $\theta < 1$, $m^2 v \leq \frac{(1+\theta)^3}{(1-\theta)\theta}$; and when $\theta > 1$, $m^2 v \leq \frac{(1+\theta)^3}{(\theta-1)\theta}$. Note $\lim_{\theta\to 0} \frac{(1+\theta)^3}{(1-\theta)\theta} = \infty$, and $\lim_{\theta\to\infty} \frac{(1+\theta)^3}{(\theta-1)\theta} = \infty$. That is, for θ close enough to 0 and ∞, the SOCs will hold for a large class of m and v. When $\frac{\partial g_1}{\partial e_1} < 0$ and $\frac{\partial g_2}{\partial e_2} < 0$ hold for $\forall e_1, e_2 > 0$ the two SOCs will ensure that (e_1^*, e_2^*) is a Nash equilibrium.

When $\frac{\partial g_1}{\partial e_1} < 0$ or $\frac{\partial g_2}{\partial e_2} < 0$ does not hold for $\forall e_1, e_2 > 0$, the FOCs can have multiple solutions and there can be multiple best responses for one player or the other. In such case, SOCs are not sufficient to for (e_1^*, e_2^*) to be a Nash equilibrium; neither player should want to deviate to any other feasible effort level. Recall, at 0 net marginal gains are positive for each player, regardless of the effort choice by the other player. This means that from (e_1^*, e_2^*) if a unilateral deviation to 0 is profitable for a player, there will be more such deviations for the player. Consider the following example:

Example 6 Let $m = n = 1,\ \theta = 1,\ v = 16$. In this case, the only pair that solves the FOC's is $(e_1^*, e_2^*) = (v/4, v/4)$ which gives $p = 1/2$.

At (e_1^*, e_2^*) payoff of the first player is $\pi_1(e_1^*, e_2^*) = \frac{1}{2}v - \frac{1}{2}(\frac{v}{(2)^2})^2 = \frac{1}{2}v(1 - \frac{v}{16})$. Given that player 2 is choosing e_2^*, does choosing $e_1 = 0$ make it better off? Yes it does. Note $\pi_1(0, e_2^*) = \frac{1}{1+exp(v/4)}v$. It can be checked that for $v = 16$, $\frac{1}{2}v(1 - \frac{v}{16}) < \frac{1}{1+exp(v/4)}v$. That is, the only solution pair emerging from FOCs is not a Nash equilibrium. Moreover, when player 2 is choosing e_2^*, other than e_1^* there are two more solution to the FOC1 one of them gives higher payoff to Player 1.

So, for (e_1^*, e_2^*) to be a Nash equilibrium a necessary condition is that a unilateral deviation from (e_1^*, e_2^*) to 0 should not profitable for either player. Let,

$\pi_1(e_1^*, e_2^*)$ and $\pi_2(e_1^*, e_2^*)$ denote the equilibrium payoff of player 1 and 2, respectively. So, for (e_1^*, e_2^*) to be a Nash equilibrium we need to ensure:

$$\begin{aligned}
\pi_1(e_1^*, e_2^*) - \pi_1(0, e_2^*) &= \frac{\theta}{(1+\theta)}v\left(1 - \frac{m^2 v\theta}{2(1+\theta)^3}\right) - \frac{\theta}{\theta + \exp(m^2 v\theta/(1+\theta)^2)}v \\
&= (\Delta\Pi_1)v \geq 0
\end{aligned} \tag{22}$$

and

$$\begin{aligned}
\pi_2(e_1^*, e_2^*) - \pi_2(e_1^*, 0) &= \frac{1}{(1+\theta)}v\left(1 - \frac{m^2 v\theta^2}{2(1+\theta)^3}\right) - \frac{1}{\theta\exp(m^2 v\theta/(1+\theta)^2) + 1}v \\
&= (\Delta\Pi_2)v \geq 0
\end{aligned} \tag{23}$$

where

$$\Delta\Pi_1 = \frac{\theta}{(1+\theta)}\left(1 - \frac{m^2 v\theta}{2(1+\theta)^3}\right) - \frac{\theta}{\theta + \exp(m^2 v\theta/(1+\theta)^2)} \tag{24}$$

and

$$\Delta\Pi_2 = \frac{1}{(1+\theta)}\left(1 - \frac{m^2 v\theta^2}{2(1+\theta)^3}\right) - \frac{1}{\theta\exp(m^2 v\theta/(1+\theta)^2)+1} \tag{25}$$

When $v > 0$, $\Delta\Pi_1 \geq 0 \iff \pi_1(e_1^*, e_2^*) - \pi_1(0, e_2^*) \geq 0$ and $\Delta\Pi_2 \geq 0 \iff \pi_2(e_1^*, e_2^*) - \pi_2(e_1^*, 0) \geq 0$. Clearly, at $v = 0$, $\Delta\Pi_1 = 0 = \Delta\Pi_2 = 0$. And for $\Delta\Pi_1 = \Delta\Pi_2 = 0$ since $e_1^* = e_2^* = 0$. It can be seen that $\lim_{v\to 0}\frac{\partial(\Delta\Pi_1)}{\partial v} = \frac{m^2\theta^2}{2(1+\theta)^4} > 0$ and $\lim_{v\to 0}\frac{\partial(\Delta\Pi_2)}{\partial v} = \frac{m^2\theta^2}{2(1+\theta)^4} > 0$, i.e., as v increases from 0, $\Delta\Pi_1$ and $\Delta\Pi_2$ increase initially to take positive values. However, as $v \to \infty$, $\Delta\Pi_i \to -\infty$. Since $\Delta\Pi_1$ and $\Delta\Pi_2$ are continuous functions of v, we know that there will exist v where $\Delta\Pi_i$ will reach zero. We can conclude that there exist $\hat{v}_1 > 0$ and $\hat{v}_2 > 0$ such that for $0 < v \leq \hat{v}_1$, $\Delta\Pi_1 \geq 0$ and for $0 < v \leq \hat{v}_2$, $\Delta\Pi_2 \geq 0$. Thus for $v \leq \hat{v} = \min\{\hat{v}_1, \hat{v}_2\}$, $\pi_i(e_i^*, e_j^*) \geq \pi_i(0, e_j^*)$ will be satisfied for $i, j = 1, 2;\ j \neq i$.

Finally, it should be noted that the above arguments apply to any strictly convex cost function. In other words, we have the following result.

Proposition 4 *For any $p \in \mathbb{P}_{\mathbb{E}}$, for any given $m = n$*

(i) If $\theta = 1$, then a unique and symmetric pure strategy Nash equilibrium exists for a range of prize value. Equilibrium probability of win $p^* = 1/2$.

(ii) If $\theta \neq 1$, then a unique and symmetric pure strategy Nash equilibrium exists for a range of prize values. Equilibrium probability of win, i.e., $p^* = \frac{\theta}{\theta+1}$.

That is, when $m = n$, the equilibrium efforts are symmetric even in the presence of asymmetry $\theta \neq 1$. Moreover, the natural advantage the players participate persists in the equilibrium.

Next consider the case where $m \neq n$. Now, for any given m, n and v, if the pair (e_1^*, e_2^*), which solves both the FOCs simultaneously, is such that $p^* = p(e_1^*, e_2^*) = 1/2$, then both SOC1 and SOC2 are trivially satisfied. When $p^* = p(e_1^*, e_2^*) > 1/2$, SOC1 is satisfied for all values of m and v, however, for SOC2 to be satisfied, we need: $\left.\frac{\partial g_2}{\partial e_2}\right|_{e_1^*, e_2^*} = n^2(2p^*-1)p^*(1-p^*)v - \psi'(e_2^*) \leq 0$. When $\psi_i(e_i) = e_i^2/2\ \ i = 1, 2$, the SOC2 holds if $n^2 v < 1/[(2p^* - 1)p^*(1 - p^*)]$. The minimum value of $\frac{1}{(2p^*-1)p^*(1-p^*)}$ is $6\sqrt{3}$ which is attained when $p^* = \frac{1}{2} + \frac{1}{2\sqrt{3}} \approx 0.8$. For any other value of $p^* > 1/2$ such as $p^* = 0.7$ or $p^* = 0.9$, $n^2 v < 6\sqrt{3}$ is still sufficient for SOC2 to be satisfied. This implies that for any given value of n, a small enough v will ensure that SOC2 is satisfied. Symmetric arguments apply when $p^* = p(e_1^*, e_2^*) < 1/2$. In this case SOC2 is trivially satisfied for all values of n, v and SOC1 will be satisfied for sufficiently low prize value. SOC1 is satisfied if $m^2 v < 1/[(1-2p^*)p^*(1-p^*)]$. The minimum value of $\frac{1}{(1-2p^*)p^*(1-p^*)}$ is also $6\sqrt{3}$ as in the previous case. It is attained when $p^* = \frac{1}{2} - \frac{1}{2\sqrt{3}} \approx 0.2$. For any other value of $p^* < 1/2$ such as $p^* = 0.1$ or $p^* = 0.3$, $m^2 v < 6\sqrt{3}$ is still sufficient for SOC1 to be satisfied. We have already seen that these conditions are sufficient to ensure that a solution to the FOCs exist. Also, as in case of $m = n$, it can be shown that for small prize values, a deviation to zero effort is not profitable for either party. Thus we can conclude that:

Proposition 5 *For any $p \in \mathbb{P}_{\mathbb{E}}$, given m, n s.t. $m \neq n$, an interior pure strategy Nash equilibrium exists for a range of (small) prize values.*

The above propositions suggest that the existence of an equilibrium in an asymmetric contest when $p \in \mathbb{P}_E$ depends on the values of m, n and v. As an illustration, consider the cases when $m = n = 1$ and with the case when $m = n = 0.1$. It can be deduced from the conditions for existence that in the later case, equilibrium exists for a wider range of prize values. In general, smaller values of m and n allow an equilibrium to exist with the wider range of prize values. Also note that as pointed out earlier, fixing $n = 1$ takes away this flexibility and thus costs the generality of the argument.

The above arguments apply to any strictly convex of effort function. Therefore, we conjecture that Proposition 5 holds more generally.

4.2 Baik's Difference Form CPF

The problems faced by the two players here are

$$\max_{e_1} \{vf(d) - \psi_1(e_1)\} \tag{26}$$

$$\max_{e_2} \{v(1 - f(d)) - \psi_2(e_2)\} \tag{27}$$

where $d = me_1 - e_2$.[19] FOCs for the above problems are given by:

$$FOC1 - \qquad g_1^B(m, v, e_1, e_2) = mvf'(d) - \psi_1'(e_1) = 0$$

$$FOC2 - \qquad g_2^B(m, v, e_1, e_2) = vf'(d) - \psi_2'(e_2) = 0$$

Since $f'(d) > 0 \ \forall d$, $e_1 = 0$ cannot solve FOC1 for any $e_2 \geq 0$ and $e_2 = 0$ cannot solve FOC2 for any $e_1 \geq 0$. Thus we can focus on the case $e_1, e_2 > 0$. From the FOCs in an equilibrium we have

$$\frac{\psi_1'(e_1^*)}{m} = \psi_2'(e_2^*)$$

If we have $\psi_i(.) = \psi(.) \ \forall i = 1, 2$ then it follows that a Nash equilibrium is symmetric iff $m = 1$, i.e. iff the contest is symmetric.

[19] We have already shown the existence of a solution for $p \in \mathbb{P}_E$ case. Combined with the fact that $\mathbb{P}_E \cap \mathbb{P}_D \neq \emptyset$, we can conclude that solution to the problem exists for at least a subclass of $p \in \mathbb{P}_D$.

From FOC1, for given m and v, $g_1^B(.)$ is a continuously differentiable function[20] and $\frac{\partial g_1^B}{\partial e_1} = m^2 v f''(d) - \psi_1''(e_1)$. Since $\psi_1''(e_1) > 0$ $\forall e_1 > 0$, for sufficiently small values of $m^2 v$, $\frac{\partial g_1^B}{\partial e_1} < 0$ $\forall e_1, e_2 > 0$. Thus, there exists a continuous (smooth) function $e_1^*(e_2)$ such that $g_1(m, v, e_1^*(e_2), e_2) = 0$.[21] In fact, in view of our assumptions, arguing as in the case of exponential form, it can be seen that there exists a pair (e_1^*, e_2^*) that solves the FOCs. Moreover, (e_1^*, e_2^*) is such that:

$$\psi_1'(e_1^*) = m\psi_2'(e_2^*) \tag{28}$$

When $\psi_i(e) = \psi(e)$, $i = 1, 2$, and $m = 1$ from (28) we can see that $e_1^* = e_2^* = \psi'^{-1}(vf'(0)) > 0$. The SOC's are:

$$SOC1 - \qquad vm^2 f''(d^*) - \psi''(e_1^*) \leq 0$$

$$SOC2 - \qquad -vf''(d^*) - \psi''(e_2^*) \leq 0$$

When $m = 1$, $d^* = 0$ which implies $f''(d^*) = 0$. Thus, the SOC's are trivially satisfied in this case. This proves our next claim.[22]

Proposition 6 *For any $p \in \mathbb{P}_D$, if $m = 1$, then there exists an interior pure strategy Nash equilibrium.*

Next consider the case $m \neq 1$. For simplicity, assume $\psi_i(e_i) = e_i^2/2$, $i = 1, 2$. Then from (28), $e_1^* = me_2^* \implies d^* = me_1^* - e_2^* = \frac{(m^2-1)}{m}e_1^*$ where $e_1^* > 0$. Thus $d^* \lesseqgtr 0$ as $m \lesseqgtr 1$. Also, given continuity of $f''(d)$ in d, it is easy to see that $f''(d^*)$ is a continuous function of m. First, the case when $m > 1$. i.e. $f''(d^*) < 0$. At (e_1^*, e_2^*) SOC1 is satisfied but SOC2 may not be. For SOC2 to be satisfied we need $f''(d^*) \geq -1/v$. Similarly, for $m < 1$, SOC2 will be trivially satisfied and SOC1 will be satisfied when $f''(d^*) \leq 1/m^2 v$. However, for sufficiently small v both SOCs will hold.

Alternatively, we can take v to be given and ask if for some possible values of m, a Nash equilibrium exists. First consider the case where $m > 1$. For given v, if $f''(.)$ is such that $f''(d^*) < -1/v$ does not hold for any possible value of m. In this case, SOC2 will be satisfied $\forall$ m s.t. $m > 1$. Therefore, let $\exists$ m such that $f''(d^*) < -1/v$. Suppose m_1 is the smallest m for which SOC2 is violated. Clearly $m_1 > 1$. Since $f''(d^*) = 0$ when $m = 1$, therefore continuity of $f''(d^*)$ implies that for $m \in (1, m_1)$ SOC2 will be satisfied. That is, $\exists$ $\delta_2(v) > 0$ such that (e_1^*, e_2^*) is a Nash equilibrium if $m \in (1, 1 + \delta_2(v))$, where $\delta_2(v) = m_1 - 1$. Similarly when $m < 1$, we can find a $\delta_1(v) > 0$ s.t. SOC1 is satisfied for $m \in (1 - \delta_1(v), 1)$. Thus,

[20] By assumption $f(d)$ is twice continuously differentiable.

[21] using Lemma 2 from Zhang & Ge (2006).

[22] Ideally one should also check that deviation to 0 effort are not profitable which is not possible without knowing more about the function $f(.)$.

Proposition 7 *Take any $p \in \mathbb{P}_D$. i) For any given m there exists $\tilde{v}(m)$ s.t. for any $v \leq \tilde{v}(m)$ there exists an interior pure strategy Nash equilibrium; ii) for any given v there exists $\tilde{v}(m)$ $\delta_1(v), \delta_2(v) > 0$ s.t. for any $m \in (1-\delta_1(v), 1+\delta_2(v))$ there exists an interior pure strategy Nash equilibrium.*

In the next section we describe comparative static results. Here we concentrate on $p \in \mathbb{P}_R$ or $p \in \mathbb{P}_E$ and compare the results for the two subclasses of logit form. We look at the impact of change in productivity, natural advantage and prize value on equilibrium efforts, cost of efforts and probability of win.

5 Comparative Statics

In this section we look at the class of CPFs $\bar{\mathbb{P}}_L = \mathbb{P}_E \cup \mathbb{P}_R$ and convex cost of effort function $\psi(.)$ for both players. Suppose $p \in \mathbb{P}_E$. On derivating g_1 and g_2 as given in (14) and (16) w.r.t α where $\alpha \in \{v, m, \theta\}$ and solving the system of equations obtained, we get

$$\frac{\partial e_1^*}{\partial \alpha} = \frac{\frac{\partial g_1}{\partial \alpha}\frac{\partial g_2}{\partial e_2} - \frac{\partial g_2}{\partial \alpha}\frac{\partial g_1}{\partial e_2}}{\frac{\partial g_1}{\partial e_2}\frac{\partial g_2}{\partial e_1} - \frac{\partial g_2}{\partial e_2}\frac{\partial g_1}{\partial e_1}} \tag{29}$$

$$\frac{\partial e_2^*}{\partial \alpha} = \frac{\frac{\partial g_2}{\partial \alpha}\frac{\partial g_1}{\partial e_1} - \frac{\partial g_1}{\partial \alpha}\frac{\partial g_2}{\partial e_1}}{\frac{\partial g_1}{\partial e_2}\frac{\partial g_2}{\partial e_1} - \frac{\partial g_2}{\partial e_2}\frac{\partial g_1}{\partial e_1}} \tag{30}$$

Let $\frac{\partial g_1}{\partial e_2}\frac{\partial g_2}{\partial e_1} - \frac{\partial g_2}{\partial e_2}\frac{\partial g_1}{\partial e_1} = A$. It is easy to check that $A < 0$ at any equilibrium. Additionally,

$$\frac{dp^*}{d\alpha} = \frac{\partial p}{\partial \alpha}\Big|_{e_1^*, e_2^*} + \frac{\partial p}{\partial e_1}\Big|_{e_1^*, e_2^*}\frac{\partial e_1^*}{\partial \alpha} + \frac{\partial p}{\partial e_2}\Big|_{e_1^*, e_2^*}\frac{\partial e_2^*}{\partial \alpha},$$

which can be expressed as:

$$\frac{dp^*}{d\alpha} = \frac{\partial p}{\partial \alpha}\Big|_{e_1^*, e_2^*} + p^*(1-p^*)\left(m\frac{\partial e_1^*}{\partial \alpha} - n\frac{\partial e_2^*}{\partial \alpha}\right). \tag{31}$$

5.1 Impact of change in value of prize v

Here $\alpha = v$. The required partials[23] in (29) and (30) are $\frac{\partial g_1}{\partial v} = mp^*(1-p^*) > 0$; $\frac{\partial g_2}{\partial v} = np^*(1-p^*) > 0$; $\frac{\partial g_1}{\partial e_1} = m^2 v(1-2p^*)p^*(1-p^*) - \psi''(e_1^*)$; $\frac{\partial g_1}{\partial e_2} = mnv(2p^*-1)p^*(1-p^*)$; $\frac{\partial g_2}{\partial e_2} = n^2 v(2p^*-1)p^*(1-p^*) - \psi''(e_2^*)$; $\frac{\partial g_2}{\partial e_1} = mnv(1-2p^*)p^*(1-p^*)$.
Some useful relations are: $\frac{\partial g_1}{\partial e_1} = \frac{m}{n}\frac{\partial g_2}{\partial e_1} - \psi''(e_1^*)$, $\frac{\partial g_2}{\partial e_2} = \frac{n}{m}\frac{\partial g_1}{\partial e_2} - \psi''(e_2^*)$, and $\frac{\partial g_1}{\partial v} = \frac{m}{n}\frac{\partial g_2}{\partial v}$.

[23] Note that the above partials are all calculated at the equilibrium (e_1^*, e_2^*), e.g. $\frac{\partial g_i}{\partial e_j}$ is nothing but $\frac{\partial g_i}{\partial e_j}\Big|_{e_1^*, e_2^*}$.

Plugging the expressions of the calculated partials[24] in (29) and (30) we find that for any Nash equilibrium, $\frac{\partial e_1^*}{\partial v} > 0$ and $\frac{\partial e_2^*}{\partial v} > 0$ and therefore $\frac{\partial (e_1^*+e_2^*)}{\partial v} > 0$.

Furthermore, we have $\frac{\partial p}{\partial v} = 0$. So, $\frac{dp^*}{dv} = \frac{\partial p^*}{\partial e_1^*}\frac{\partial e_1^*}{\partial v} + \frac{\partial p^*}{\partial e_2^*}\frac{\partial e_2^*}{\partial v}$, which can be rewritten as

$$\frac{dp^*}{dv} = p^*(1-p^*)\big(m\frac{\partial e_1^*}{\partial v} - n\frac{\partial e_2^*}{\partial v}\big).$$

where

$$m\frac{\partial e_1^*}{\partial v} - n\frac{\partial e_2^*}{\partial v} = \frac{-p^*(1-p^*)\big(m^2\psi''(e_2^*) - n^2\psi''(e_1^*)\big)}{A}$$

For the cost function $\frac{e_1^2}{2}$ we get $m\frac{\partial e_1^*}{\partial v} - n\frac{\partial e_2^*}{\partial v} = \frac{-p^*(1-p^*)(m^2-n^2)}{A}$. Thus, $m\frac{\partial e_1^*}{\partial v} - n\frac{\partial e_2^*}{\partial v} \gtrless 0$ as $m \gtrless n$. Therefore,

$$\frac{dp^*}{dv} = p^*(1-p^*)\big(m\frac{\partial e_1^*}{\partial v} - n\frac{\partial e_2^*}{\partial v}\big) \gtrless 0 \ as \ m \gtrless n$$

These results also hold for any $p \in \mathbb{P}_R$ with convex costs. Thus we get our next result.

Proposition 8 For $p \in \bar{\mathbb{P}}_L$, for any given value of θ, m and n, ceteris paribus

(i) the equilibrium effort levels for both players (and thus the total effort expended in the contest) increases with the value of prize.

(ii) the equilibrium probability of win for player 1 increases(decreases) with the value of prize if $m > (<)\ n$.

5.2 Impact of change in natural advantage, (θ)

When $\alpha = \theta$, we have $\frac{\partial g_1}{\partial \theta} = mv(1-2p^*)\frac{\partial p}{\partial \theta}\Big|_{e_1^*,e_2^*}$; $\frac{\partial g_2}{\partial \theta} = nv(1-2p^*)\frac{\partial p}{\partial \theta}\Big|_{e_1^*,e_2^*}$, where

$$\frac{\partial p}{\partial \theta}\Big|_{e_1^*,e_2^*} = \frac{\exp(me_1^*)\exp(ne_2^*)}{(\theta\exp(me_1^*) + \exp(ne_2^*))^2} = \frac{p^*(1-p^*)}{\theta} > 0.$$

For this case, $\frac{\partial g_1}{\partial \theta} = \frac{m}{n}\frac{\partial g_2}{\partial \theta}$. The other partials are the same as in the previous case. The effect of θ depends on the value of the equilibrium value of p, i.e., p^*.

$$\frac{\partial e_1^*}{\partial \theta} = \frac{\frac{\partial g_1}{\partial \theta}\big(-\psi''(e_2^*)\big)}{A} = \frac{-mv(1-2p^*)\frac{\partial p^*}{\partial \theta}\psi''(e_2^*)}{A} \tag{32}$$

$$\frac{\partial e_2^*}{\partial \theta} = \frac{\frac{\partial g_2}{\partial \theta}\big(-\psi''(e_1^*)\big)}{A} = \frac{-nv(1-2p^*)\frac{\partial p^*}{\partial \theta}\psi''(e_1^*)}{A} \tag{33}$$

[24]It is easy to check from the expressions that $\frac{\partial g_1}{\partial e_2}\frac{\partial g_2}{\partial e_1} \leq 0$. Also, at any Nash equilibrium $\frac{\partial g_1}{\partial e_1}\frac{\partial g_2}{\partial e_2} \geq 0$.

It is easy to see that $\frac{\partial e_1^*}{\partial \theta} \gtreqless 0 \; as \; p^* \lesseqgtr 1/2$. Same is true for $\frac{\partial e_2^*}{\partial \theta}$, i.e. $\frac{\partial e_2^*}{\partial \theta} \gtreqless 0 \; as \; p^* \lesseqgtr 1/2$. e_1^* and e_2^* are maximum when $p^* = 1/2$ i.e. when the contest is symmetric.

For instance, when $m = n$, an equilibrium say $(e_1^*, e_2^*, p^*) = (\frac{mv\theta}{(\theta+1)^2}, \frac{mv\theta}{(\theta+1)^2}, \frac{\theta}{\theta+1})$. In this case, if $\theta < 1$ then $p^* < 1/2$ and hence $\frac{\partial e_1^*}{\partial \theta} > 0$ and $\frac{\partial e_2^*}{\partial \theta} > 0$. The opposite holds true for $\theta > 1$. The closer θ is to 1, i.e., the more symmetric the contest is, higher are the efforts expended. [25]

However, the equilibrium probability of win, p^*, increases with θ, i.e., $\frac{dp^*}{d\theta} > 0$ holds.

$$
\begin{aligned}
\frac{dp^*}{d\theta} &= \frac{\partial p^*}{\partial \theta} + \frac{\partial p^*}{\partial e_1^*}\frac{\partial e_1^*}{\partial \theta} + \frac{\partial p^*}{\partial e_2^*}\frac{\partial e_2^*}{\partial \theta} \\
&= \frac{\partial p^*}{\partial \theta} + p^*(1-p^*)\big(m\frac{\partial e_1^*}{\partial \theta} - n\frac{\partial e_2^*}{\partial \theta}\big) \\
&= \frac{\partial p^*}{\partial \theta}\Big[1 - \frac{p^*(1-p^*)(1-2p^*)v(m^2\psi''(e_2^*) - n^2\psi''(e_1^*))}{A}\Big]
\end{aligned}
$$

It turns out that $p^*(1-p^*)(1-2p^*)v(m^2\psi''(e_2^*) - n^2\psi''(e_1^*)) > A$, and since $A < 0$ at any Nash equilibrium, therefore $\frac{dp^*}{d\theta} > 0$. The above findings can be summarized as the next two results.

Proposition 9 *For $p \in \bar{\mathbb{P}}_L$, ceteris paribus, for any given given $m, n > 0$*

(i) when $m = n$, probability of win for a player is a monotonically increasing function of her relative natural advantage.

(ii) the effort levels increases(decreases) with increase in θ if $p^* < 1/2$ ($p^* > 1/2$). The maximum is achieved at $p^* = 1/2$ i.e. when the contest becomes more symmetric.

Proposition 10 *For $p \in \mathbb{P}_E$ and any given m, n, ceteris paribus, probability of win for a player is a monotonically increasing function of her relative natural advantage.*

5.3 Effect of change in m

When $\alpha = m$, the required partials are $\frac{\partial g_1}{\partial m} = p^*(1-p^*)v + m(1-2p^*)v\frac{\partial p}{\partial m}\big|_{e_1^*, e_2^*}$; $\frac{\partial g_2}{\partial m} = n(1-2p^*)v\frac{\partial p}{\partial m}\big|_{e_1^*, e_2^*}$. Moreover, it can be seen that $\frac{\partial p}{\partial m} > 0$ and $\frac{\partial g_1}{\partial m} = p(1-p)v + \frac{m}{n}\frac{\partial g_2}{\partial m}$. The other partials are the same as in the previous case.

[25] This result is parallel to the result given by Snyder (1989) with ratio form contest success function and linear costs and result in Baik (1994).

Using (29), (30) and the above partials, we get

$$\frac{\partial e_1^*}{\partial m} = \frac{p^*(1-p^*)v\frac{\partial g_2}{\partial e_2} - m(1-2p^*)v\frac{\partial p^*}{\partial m}}{A} \tag{34}$$

$$\frac{\partial e_2^*}{\partial m} = \frac{-\big(p^*(1-p^*)v\frac{\partial g_2}{\partial e_1} + n(1-2p^*)v\frac{\partial p^*}{\partial m}\big)}{A} \tag{35}$$

For given θ, n and v, it can be seen that: If $p^* < 1/2$, then $\frac{\partial e_2^*}{\partial m} > 0$ and $\frac{\partial e_1^*}{\partial m} > 0$. On the other hand, if $p^* > 1/2$ then $\frac{\partial e_2^*}{\partial m} < 0$, however $\frac{\partial e_1^*}{\partial m} \lesseqgtr 0$ can hold. That is, the effect of m on e^* is ambiguous.
Furthermore,

$$\frac{dp^*}{dm} = \frac{\partial p^*}{\partial m} + p^*(1-p^*)(m\frac{\partial e_1^*}{\partial m} - n\frac{\partial e_2^*}{\partial m}).$$

Since $m\frac{\partial e_1^*}{\partial m} - n\frac{\partial e_2^*}{\partial m} \lesseqgtr 0$ can hold, the sign of $\frac{dp^*}{dm}$ is ambiguous.

It should be noted that an increase in m or θ represent a favorable change for player 1. However, our results show that these changes may or may not have the similar impact. To sum up, given m, n, θ has a positive monotonic impact on this probability. However, given for θ and n, m has an ambiguous impact on the probability of win for player 1. Equilibrium effort for player 1 is a non-monotonic function of its productivity m as well as θ, ceteris paribus. For given value of m and n, θ only effects the level of impact of prize value on the probability of win. However, whether p^* increases or decreases with change in v depends on whether $m \gtreqless n$.

5.4 $\mathbb{P}_E$ Versus $\mathbb{P}_R$

The above results on the comparative static for $p \in \mathbb{P}_E$ are very similar to those for $p \in \mathbb{P}_R$ found in the existing literature. For instance, in the existing literature as well as in our paper, *ceteris paribus*, the individual and hence the total effort provided by the players are increasing function of the value of the prize. Consequently, the total cost of efforts incurred by the players is also increasing function of the value of the prize. Additionally, assuming $m = n$ for $p \in \mathbb{P}_E$ as well as for $p \in \mathbb{P}_R$, the equilibrium probability of win for the first player increases with the player's natural advantage, i.e., θ. Moreover, the effort levels increase as the contest becomes more symmetric i.e. as $\theta \to 1$.

It should be kept in mind that in the literature the results for the ratio-form are derived using linear costs. In contrast, we have worked here with convex cost of effort functions. However, for the ratio form CPFs even when the linear cost is replaced with a convex cost function, the nature of optimization problem essentially remains the same. Formally, from (12) it can be checked that when $p \in \mathbb{P}_R$, we can rewrite player i's optimization problem as

$$\max_{x_i} \left\{ \frac{\theta(y_i(x_i))^m}{\theta(y_i(x_i))^m + (y_j(x_j))^n} v - x_i \right\}$$

where $x_i = \psi_i(e_i)$, and $y_i(x_i) = y_i(\psi_i(e_i)) = e_i$.[26] Since the nature of the optimization problems does not change, results with strict convex cost functions are very similar. It should be kept in mind that the claims in the literature related to the effort levels will correspond to the levels of costs in our framework. Working backward we can derive the results for the levels of effort.[27]

6 Concluding Remarks

By assuming strictly convex cost of effort functions, we have demonstrated the existence of pure strategy Nash equilibrium for the difference form CPFs, thus breaking down the 'impossibility theorem' given by Hirshleifer (1989). Moreover, we have shown that properties of the equilibria and the comparative statics for the difference form CPFs closely resemble those for the ratio form. However, there are a few interesting differences as well. When contests are asymmetric with respect to innate ability or productivity, unlike the ratio-form, in this case the existence of pure strategy Nash equilibrium is sensitive to the size of the prize.

However, with convex cost functions, some interesting differences emerge between the ratio form and the logit form CPFs. For illustration, consider the case when $m = n$ and $\psi(e) = e^2/2$: For $p \in \mathbb{P}_E$, we have $e_1^* = e_2^* = \frac{mv\theta}{(\theta+1)^2}$ and $\psi(e_1^*) = \psi(e_2^*) = \frac{1}{2}\left(\frac{mv\theta}{(\theta+1)^2}\right)^2$. However, for $p \in \mathbb{P}_R$, it can be seen that the equilibrium values of costs are $x_1^{r*} = x_2^{r*} = \frac{mv\theta}{(\theta+1)^2}$ and the corresponding levels of effort levels are $e_1^{r*} = e_2^{r*} = \left(\frac{2mv\theta}{(\theta+1)^2}\right)^{\frac{1}{2}}$.

Therefore, when $p \in \mathbb{P}_R$, the cost of effort is a constant proportion of the prize value and given by $\frac{m\theta}{(\theta+1)^2}$. That is, the proportion of prize value spent in the form of cost of efforts does not change with the value of the prize. On the other hand, for $p \in \mathbb{P}_E$, the proportion of prize value expanded as the cost of total effort is $\left(\frac{m\theta}{(\theta+1)^2}\right)^2 v$, which increases in direct proportion to the value of prize. As to the levels of efforts, when $p \in \mathbb{P}_E$, the effort levels, $\frac{2m\theta}{(\theta+1)^2}$, are a constant proportion of the prize value. However, for the ratio form the effort levels are inversely proportional to the prize value. Moreover, if we suppose $\theta = 1$, then $e_1^* + e_2^* = \frac{mv}{2}$ and $e_1^{r*} + e_2^{r*} = 2\left(\frac{2mv}{4}\right)^{\frac{1}{2}}$.

$$e_1^* + e_2^* \gtreqless e_1^{r*} + e_2^{r*} \ \ as \ \ mv \gtreqless 8$$

i.e. either of the two CPFs can lead to greater total effort in the equilibrium.

[26] For the ratio form CPFs we have used cost function of the specific form $\psi = e^k$, $k > 1$ because it allows us to keep expressions same as normal ratio form!

[27] The details are are presented as Web-index and are available at http://econdse.org/ram-research/.

References

Abbink, K., Brandts, J., Herrmann, B. & Orzen, H. (2010), ‘Intergroup conflict and intra-group punishment in an experimental contest game’, *The American Economic Review* **100**(1), 420–447.

Abbink, K., Brandts, J., Herrmann, B. & Orzen, H. (2012), ‘Parochial altruism in inter-group conflicts’, *Economics Letters* **117**(1), 45–48.

Ahn, T., Isaac, R. M. & Salmon, T. C. (2011), ‘Rent seeking in groups’, *International Journal of Industrial Organization* **29**(1), 116–125.

Anderton, C. H. (2000), ‘An insecure economy under ratio and logistic conflict technologies’, *Journal of conflict Resolution* **44**(6), 823–838.

Baik, K. H. (1994), ‘Effort levels in contests with two asymmetric players’, *Southern Economic Journal* **61**(2), 367–378.

Baik, K. H. (1998), ‘Difference-form contest success functions and effort levels in contests’, *European Journal of Political Economy* **14**(4), 685–701.

Baik, K. H. (2004), ‘Two-player asymmetric contests with ratio-form contest success functions’, *Economic Inquiry* **42**(4), 679–689.

Baron, D. P. (1994), ‘Electoral competition with informed and uninformed voters.’, *American Political Science Review* **88**(01), 33–47.

Baye, M. R. & Hoppe, H. C. (2003), ‘The strategic equivalence of rent-seeking, innovation, and patent-race games’, *Games and economic behavior* **44**(2), 217–226.

Baye, M. R., Kovenock, D. & De Vries, C. G. (1994), ‘The solution to the tullock rent-seeking game when $r > 2$: Mixed-strategy equilibria and mean dissipation rates’, *Public Choice* **81**(3-4), 363–380.

Beviá, C. & Corchón, L. C. (2015), ‘Relative difference contest success function’, *Theory and Decision* **78**(3), 377–398.

Caruso, R. (2006), ‘A trade institution as a peaceful institution? a contribution to integrative theory’, *Conflict Management and Peace Science* **23**(1), 53–72.

Caruso, R. (2007), ‘Continuing conflict and stalemate: a note’, *Available at SSRN 975023* .

Che, Y.-K. & Gale, I. (2000), ‘Difference-form contests and the robustness of all-pay auctions’, *Games and Economic Behavior* **30**(1), 22–43.

Chowdhury, S. M. & Sheremeta, R. M. (2011), ‘Multiple equilibria in tullock contests’, *Economics Letters* **112**(2), 216–219.

Clark, D. J. & Riis, C. (1998), ‘Contest success functions: an extension’, *Economic Theory* **11**(1), 201–204.

Corchón, L. & Dahm, M. (2010), ‘Foundations for contest success functions’, *Economic Theory* **43**(1), 81–98.

Cornes, R. & Hartley, R. (2005), ‘Asymmetric contests with general technologies’, *Economic theory* **26**(4), 923–946.

Cornes, R. & Hartley, R. (2012), ‘Well-behaved aggregative games’, *Working Paper, Australian National University* .

Coughlin, P. J. (1990), ‘Candidate uncertainty and electoral equilibria’, *Advances in the spatial theory of voting* **145**, 166.

Cubel, M. (2014), ‘Fiscal equalization and political conflict’, *The Economics of Conflict: Theory and Empirical Evidence* p. 23.

Dahm, M., Porteiro, N. et al. (2005), A micro-foundation for non-deterministic contests of the logit form, Technical report.

Demougin, D. & Fluet, C. (2003), ‘Inequity aversion in tournaments’.

Dixit, A. (1987), ‘Strategic behavior in contests’, *The American Economic Review* pp. 891–898.

Einy, E., Haimanko, O., Moreno, D., Sela, A. & Shitovitz, B. (2013), ‘Tullock contests with asymmetric information’.

Esteban, J. & Sákovics, J. (2003), ‘Olson vs. coase: Coalitional worth in conflict’, *Theory and Decision* **55**(4), 339–357.

Ewerhart, C. (2015), ‘Contest success functions: the common-pool perspective’, *University of Zurich, Department of Economics, Working Paper* (195).

Ewerhart, C. & Quartieri, F. (2015), ‘Unique equilibrium in incomplete information contests with budget constraints’, *University of Zurich, Department of Economics, Working Paper* (133).

Fey, M. (2008), ‘Rent-seeking contests with incomplete information’, *Public Choice* **135**(3-4), 225–236.

Fu, Q. & Lu, J. (2012), ‘The optimal multi-stage contest’, *Economic Theory* **51**(2), 351–382.

Fullerton, R. L. & McAfee, R. P. (1999), ‘Auctioning entry into tournaments’, *Journal of Political Economy* **107**(3), 573–605.

Gersbach, H. & Haller, H. (2009), 'The affectionate society: does competition for partners promote friendliness?', *Economic Theory* **40**(3), 389–403.

Grund, C. & Sliwka, D. (2005), 'Envy and compassion in tournaments', *Journal of Economics & Management Strategy* **14**(1), 187–207.

Harbring, C. & Irlenbusch, B. (2003), 'An experimental study on tournament design', *Labour Economics* **10**(4), 443–464.

Hillman, A. L. & Riley, J. G. (1989), 'Politically contestable rents and transfers', *Economics & Politics* **1**(1), 17–39.

Hirshleifer, J. (1989), 'Conflict and rent-seeking success functions: Ratio vs. difference models of relative success', *Public choice* **63**(2), 101–112.

Hirshleifer, J. (1991), 'The paradox of power', *Economics & Politics* **3**(3), 177–200.

Hirshleifer, J. (1995), 'Anarchy and its breakdown', *Journal of Political Economy* pp. 26–52.

Hoehn, T. & Szymanski, S. (2010), The americanization of european football, *in* 'Football Economics and Policy', Springer, pp. 52–96.

Holmstrom, B. (1982), 'Moral hazard in teams', *The Bell Journal of Economics* pp. 324–340.

Hurley, T. M. & Shogren, J. F. (1998), 'Asymmetric information contests', *European Journal of Political Economy* **14**(4), 645–665.

Hwang, S.-H. (2009), 'Contest success functions: Theory and evidence'.

Jia, H. (2008), 'A stochastic derivation of the ratio form of contest success functions', *Public Choice* **135**(3-4), 125–130.

Jia, H., Skaperdas, S. & Vaidya, S. (2013), 'Contest functions: theoretical foundations and issues in estimation', *International Journal of Industrial Organization* **31**(3), 211–222.

Konrad, K. A. (2000), 'Sabotage in rent-seeking contests', *Journal of Law, Economics, and Organization* **16**(1), 155–165.

Konrad, K. A. et al. (2009), 'Strategy and dynamics in contests', *OUP Catalogue* .

Loury, G. C. (1979), 'Market structure and innovation', *The quarterly journal of economics* pp. 395–410.

McFadden, D. et al. (1973), 'Conditional logit analysis of qualitative choice behavior'.

Moldovanu, B. & Sela, A. (2001), 'The optimal allocation of prizes in contests', *The American Economic Review* **91**(3), 542–558.

Moldovanu, B. & Sela, A. (2006), 'Contest architecture', *Journal of Economic Theory* **126**(1), 70–96.

Nitzan, S. (1991), 'Rent-seeking with non-identical sharing rules', *Public Choice* **71**(1-2), 43–50.

Nti, K. O. (1997), 'Comparative statics of contests and rent-seeking games', *International Economic Review* pp. 43–59.

Nti, K. O. (1999), 'Rent-seeking with asymmetric valuations', *Public Choice* **98**(3-4), 415–430.

Paul, C. & Wilhite, A. (1991), 'Rent-seeking, rent-defending, and rent dissipation', *Public Choice* **71**(1-2), 61–70.

Polishchuk, L. & Tonis, A. (2013), 'Endogenous contest success functions: a mechanism design approach', *Economic Theory* **52**(1), 271–297.

Reinganum, J. F. (1989), 'The timing of innovation: Research, development, and diffusion', *Handbook of industrial organization* **1**, 849–908.

Robson, A. & Skaperdas, S. (2008), 'Costly enforcement of property rights and the coase theorem', *Economic Theory* **36**(1), 109–128.

Schweinzer, P. & Segev, E. (2012), 'The optimal prize structure of symmetric tullock contests', *Public Choice* **153**(1-2), 69–82.

Skaperdas, S. (1996), 'Contest success functions', *Economic theory* **7**(2), 283–290.

Skaperdas, S. & Grofman, B. (1995), 'Modeling negative campaigning', *American Political Science Review* **89**(01), 49–61.

Skaperdas, S., Toukan, A., Vaidya, S. et al. (2013), 'Difference-form persuasion contests'.

Snyder, J. M. (1989), 'Election goals and the allocation of campaign resources', *Econometrica: Journal of the Econometric Society* pp. 637–660.

St-Pierre, M. (2016), 'The role of inequality on effort in tournaments', *Mathematical Social Sciences* **81**, 38–52.

Szidarovszky, F. & Okuguchi, K. (1997), 'On the existence and uniqueness of pure nash equilibrium in rent-seeking games', *Games and Economic Behavior* **18**(1), 135–140.

Szymanski, S. (2003), ‘The economic design of sporting contests’, *Journal of economic literature* **41**(4), 1137–1187.

Szymanski, S. & Valletti, T. M. (2005), ‘Incentive effects of second prizes’, *European Journal of Political Economy* **21**(2), 467–481.

Wärneryd, K. (2000), ‘In defense of lawyers: moral hazard as an aid to cooperation’, *Games and Economic Behavior* **33**(1), 145–158.

Wärneryd, K. (2003), ‘Information in conflicts’, *Journal of Economic Theory* **110**(1), 121–136.

Zhang, W. & Ge, S. S. (2006), ‘A global implicit function theorem without initial point and its applications to control of non-affine systems of high dimensions’, *Journal of Mathematical Analysis and Applications* **313**(1), 251–261.